

MATH 425b ASSIGNMENT 7 SOLUTIONS
 SPRING 2009
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Chapter 9:

(26) Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be given by $f(x, y) = |x|$. Then $(D_2f)(x, y) = 0$ for all x, y , so $(D_{12}f)(x, y) = 0$ for all x, y . But $(D_1f)(0, y)$ would be the derivative of $g(x) = |x|$ at $x = 0$, which doesn't exist.

(29) If we interchange i_j and i_{j+1} for some $j \geq 1$, we get

$$D_{i_1 \dots i_j i_{j+1} \dots i_k} f = D_{i_1 \dots i_{j-1}} (D_{i_j i_{j+1}} (D_{i_{j+2} \dots i_k} f)). \quad (1)$$

Since $f \in \mathcal{C}^k$, we have $D_{i_{j+2} \dots i_k} f \in \mathcal{C}^{j+1} \subset \mathcal{C}_2$ so $D_{i_j i_{j+1}} (D_{i_{j+2} \dots i_k} f) = D_{i_{j+1} i_j} (D_{i_{j+2} \dots i_k} f)$ by Theorem 9.41. As in (1),

$$\begin{aligned} D_{i_1 \dots i_{j+1} i_j \dots i_k} f &= D_{i_1 \dots i_{j-1}} (D_{i_{j+1} i_j} (D_{i_{j+2} \dots i_k} f)) \\ &= D_{i_1 \dots i_{j-1}} (D_{i_j i_{j+1}} (D_{i_{j+2} \dots i_k} f)) \\ &= D_{i_1 \dots i_j i_{j+1} \dots i_k} f, \end{aligned}$$

so we can switch adjacent indices i_j, i_{j+1} . Any permutation can be constructed out of such switches so indices can be arbitrarily permuted.

(A) Since the second derivative in y is negative, the maximizing value $g(x)$ for y must be unique for each fixed x , so it is given as the unique solution of $(D_2f)(x, y) = 0$. D_2f is a \mathcal{C}' function from an open subset of \mathbb{R}^2 to \mathbb{R} whose derivative has matrix $[A_x \ A_y]$ with $A_y = D_{22}f < 0$ by assumption. By the Implicit Function Theorem (applied with the roles of x and y interchanged), g is a \mathcal{C}' function.

(B)(a) Suppose (x, y_1) and (x, y_2) are points of U , with $y_1 < y_2$. Since U is convex, we have $(x, y) \in U$ for all $y \in (y_1, y_2)$. Therefore by the Fundamental Theorem of Calculus,

$$f(x, y_2) - f(x, y_1) = \int_{y_1}^{y_2} \frac{\partial f}{\partial y}(x, y) dy = 0.$$

This means that for fixed x , $f(x, y)$ has the same value, call it $g(x)$, for all y such that $(x, y) \in U$.

(b) Let U be the C-shaped region $((0, 2) \times (0, 3)) \setminus ([1, 2] \times [1, 2])$. Define f to be 0 on the left side $[(0, 1) \times (0, 3)]$ and on the bottom $[1, 2] \times (0, 1)$, and define $f(x, y) = (x - 1)^2$ on the top, $[1, 2] \times (2, 3)$. Since the derivatives of 0 and $(x - 1)^2$ are both 0 at $x = 1$, the

derivatives from the left side and right side match on the line $x = 1, 2 < y < 3$, so f is \mathcal{C}' . But f is not constant on any line $x = c$ with $1 < c < 2$, since $f(c, y) = 0$ for $c \in (0, 1)$ and $f(c, y) = (c - 1)^2$ for $2 < c < 3$.

(C)(a) Let $f = (f_1, f_2)$. In the notation of the Implicit Function Theorem, the matrices at $\mathbf{x} = (3, 2), \mathbf{y} = (1, 1, 2)$ are

$$A_x = \begin{bmatrix} -2x_1 & 2x_2 \\ 2x_1 & 4x_2 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 6 & 8 \end{bmatrix}, \quad A_y = \begin{bmatrix} 2y_1 & 2y_2 & 2y_3 \\ 2y_1 & -2y_2 & 2y_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 2 & -2 & 4 \end{bmatrix}.$$

Since A_x is invertible, we can locally solve for $\mathbf{x} = g(\mathbf{y})$ with

$$g'((1, 1, 2)) = -A_x^{-1}A_y = \frac{1}{72} \begin{bmatrix} 8 & -4 \\ -6 & -6 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 2 & -2 & 4 \end{bmatrix} = \frac{1}{72} \begin{bmatrix} 8 & 24 & 16 \\ -24 & 0 & -48 \end{bmatrix}.$$

This means that if we move \mathbf{y} in direction $\Delta\mathbf{y} = (0, 1, 1)$, \mathbf{x} must move in direction

$$\Delta\mathbf{x} = g'((1, 1, 2))\Delta\mathbf{y} = \frac{1}{72} \begin{bmatrix} 8 & 24 & 16 \\ -24 & 0 & -48 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/9 \\ -2/3 \end{bmatrix}.$$

(b) We need

$$f'(x, y) \begin{bmatrix} \mathbf{h} \\ \mathbf{k} \end{bmatrix} = 0,$$

that is, $A_y\mathbf{k} = -A_x\mathbf{h}$. This can be solved for \mathbf{k} if and only if $A_x\mathbf{h}$ is in the range of A_y , which is the span of the columns $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$. In other words, \mathbf{h} must be in the span of $A_x^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 1/3 \end{bmatrix}$ and $A_x^{-1} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$. But this span is two-dimensional so \mathbf{h} can be arbitrary.

(c) A_x and A_y are still given by the formulas in (a) as functions of \mathbf{x} and \mathbf{y} , which at the new values $\mathbf{x} = (3, 2), \mathbf{y} = (1, 0, 2)$ gives

$$A_x = \begin{bmatrix} -6 & 4 \\ 6 & 8 \end{bmatrix}, \quad A_y = \begin{bmatrix} 2 & 0 & 4 \\ 2 & 0 & 4 \end{bmatrix}.$$

The range of A_y now consists only of multiples of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ so \mathbf{h} must be a multiple of $A_x^{-1} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 1/3 \end{bmatrix}$.