MATH 425b ASSIGNMENT 7 (Revised)
SPRING 2009
Prof. Alexander
Due Monday April 6.

Rudin Chapter 9 #26, 29 and:

(A) This problem tests your understanding of the definition of derivative. A system has 3 inputs $x_1, x_2, x_3$, and the corresponding output $f(\mathbf{x})$ has 3 components; more precisely $f : \mathbb{R}^3 \to \mathbb{R}^3$ is $\mathcal{C}'$. The function $f$ is not known to the researcher, but he can input values $\mathbf{x}$ and observe the output values $f(\mathbf{x})$. So far he has the following observations:

$$
\begin{aligned}
f(2,3,4) &= (7,6,5) \\
f(2.01,3,4) &= (6.99,6.03,5.04) \\
f(2.01,3.01,4) &= (7.01,6.06,5.05) \\
f(2.01,3.01,4.01) &= (7.01,6.02,5)
\end{aligned}
$$

Use this information to find the approximate value of the derivative $f'(2,3,4)$ (expressed as a matrix) and to estimate $f(2,3.01,4.01)$.

(B) Let $U$ be a connected open subset of $\mathbb{R}^2$ and let $f : U \to \mathbb{R}$ be a $\mathcal{C}'$ function with $\frac{\partial f}{\partial y} = 0$ everywhere in $U$.

(a) Recall that $U$ is called *convex* if for any two points in $U$, the line segment connected these two points is contained in $U$. If $U$ is convex, show that $f$ does not depend on $y$, that is, $f(x,y) = g(x)$ for all $x, y$.

(b) In contrast to (a), find an example of a non-convex $U$ and an $f$ as above, in which $f$ does depend on $y$.

(C) Let $E \subset \mathbb{R}^2$ be open and let $f : E \to \mathbb{R}$. Suppose that $D_1 f$ exists everywhere in $E$ and is continuous at some point $\mathbf{a} \in E$, while $D_2 f$ exists only at $\mathbf{a}$. Show that $f$ is differentiable at $\mathbf{a}$. HINT: Express $f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a})$ as a sum of two increments. Also, if $f'(\mathbf{a})$ does exist its matrix must have entries $D_1 f(\mathbf{a}), D_2 f(\mathbf{a})$ so you just have to show that the transformation with this matrix satisfies the definition of derivative.

**HINTS:**

(B)(b) This is a little tricky—you have to think geometrically about the graph of such an $f$. Use a set $U$ which is actually "U" shaped (not necessarily opening upward, though.) To build your $f$, you may want to make use of functions on $\mathbb{R}$ which are $\mathcal{C}'$ and not constant, but are constant on an interval, for example,

$$
h(t) = \begin{cases} 0, & t \le 0, \\ t^2, & t > 0. \end{cases}
$$