MATH 425b ASSIGNMENT 5 SOLUTIONS
SPRING 2009
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Chapter 9:

(6) If $(x, y) \neq (0, 0)$ then

$$(D_1 f)(x, y) = \frac{\partial}{\partial x} \frac{xy}{x^2 + y^2} = \frac{(x^2 + y^2)y - xy \cdot 2x}{(x^2 + y^2)^2} = \frac{y(y^2 - x^2)}{(x^2 + y^2)^2},$$

and similarly

$$(D_2 f)(x, y) = \frac{x(x^2 - y^2)}{(x^2 + y^2)^2}.$$

Since $f(x, y) = 0$ for (x, y) on an axis (i.e. $x = 0$ or $y = 0$), we have $(D_1 f)(0, 0) = (D_2 f)(0, 0) = 0$. But $\lim_{x \to 0} f(x, x) = \lim_{x \to 0} \frac{1}{2} = \frac{1}{2}$ while $\lim_{x \to 0} f(x, 0) = 0$, and these limits being different means f is not continuous at $(0, 0)$.

(7) Suppose $E \subset \mathbb{R}$ is open, $f : E \to \mathbb{R}$, and there exists M such that $|(D_j f)(x)| \leq M$ for all $x \in E$ and all $j \leq M$. Let $x \in E$ and let B be a neighborhood of x with $B \subset E$. Suppose $x + h \in B$ for some h, and let $v_k = x + \sum_{i=1}^{k} h_i e_i$, so $v_0 = x, v_n = y$, and all $v_k \in B$. Fix some $k \leq n$. Since $D_k f$ exists in B, the function $g(t) = f(v_{k-1} + t h_k e_k)$ is differentiable in $[0, 1]$ with $g'(t) = (D_k f)(v_{k-1} + t h_k e_k)\ h_k$. By the mean value theorem, there exists $t \in (0, 1)$ such that

$$f(v_k) - f(v_{k-1}) = g(1) - g(0) = g'(t)(1 - 0),$$

so $|f(v_k) - f(v_{k-1})| \leq |g'(t)| \leq M|h_k|$. Summing, we get

$$|f(x + h) - f(x)| \leq \left| \sum_{k=1}^{n} (f(v_k) - f(v_{k-1})) \right| \leq M \sum_{k=1}^{n} |h_k|,$$

which approaches 0 as $h \to 0$, so f is continuous.

(8) If f has a local maximum, then for every i, the function $g(t) = f(x + t e_i)$ has a local maximum at $t = 0$. By Theorem 5.8, $g'(0) = 0$. But $g'(0) = (D_i f)(x)$ so $(D_i f(x) = 0$ for all i, so all entries in the $1 \times n$ matrix of $f'(x)$ are 0, that is, $f'(x) = 0$.

(13) We have $1 = |f(t)|^2 = \sum_{i=1}^{3} f_i(t)^2$ for all t, so

$$0 = \frac{d}{dt} \sum_{i=1}^{3} f_i(t)^2 = \sum_{i=1}^{3} 2 f_i(t) f_i'(t) = 2 f'(t) \cdot f(t).$$

Geometrically, if the length stays fixed (i.e. $|f(t)| = 1$ for all t) then we are moving in a circle around the origin so the velocity $f'(t)$ is perpendicular to the location $f(t)$.

(27) Let

$$f(x,y) = \frac{xy(x^2-y^2)}{x^2+y^2}, \quad (x,y) \neq (0,0), \quad \text{and } f(0,0) = 0.$$

(a) For $(x,y) \neq (0,0)$, f is continuous since the denominator x^2+y^2 is not 0. Also

$$(D_1 f)(x,y) = \frac{y(3x^2-y^2)}{x^2+y^2} - \frac{2x^2y(x^2-y^2)}{(x^2+y^2)^2} = y\left(3\frac{x^2}{x^2+y^2} - \frac{y^2}{x^2+y^2}\right) - \frac{2x^2}{x^2+y^2}\,\frac{x^2-y^2}{x^2+y^2}y,$$

which is continuous for the same reason. Further, stil for $(x,y) \neq (0,0)$, we have

$$\frac{x^2}{x^2+y^2} \leq 1, \qquad \frac{y^2}{x^2+y^2} \leq 1, \tag{1}$$

so from the formula for $D_1 f$,

$$|(D_1 f)(x,y)| \leq 4|y| + 2|y| = 6|y| \to 0 \quad \text{as } (x,y) \to (0,0).$$

Similarly, $(D_2 f)(x,y)$ in continuous at all $(x,y) \neq (0,0)$ and $(D_2 f)(x,y) \to 0$ as $(x,y) \to (0,0)$.

Now consider what happens at $(0,0)$. By (1), $|f(x,y)| \leq |xy| \to 0$ as $(x,y) \to (0,0)$, so f is continuous at $(0,0)$. Next,

$$(D_1 f)(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} 0 = 0,$$

so $D_1 f$ is continuous at $(0,0)$. Similarly $(D_2 f)(0,0) = 0$ so $D_2 f$ is continuous at $(0,0)$.

(b and c) For $(x,y) \neq (0,0$, we have $f(x,y) = -f(y,x)$ so

$$(D_2 f)(x,y) = -(D_1 f)(y,x). \tag{2}$$

Using the formula for $D_1 f$ in (a),

$$(D_{21} f)(x,y) = \frac{3x^2-y^2}{x^2+y^2} - \frac{8x^2y^2}{(x^2+y^2)^2} - \frac{(x^2+y^2)(2x^4-6x^2y^2) - 8x^2y^2(x^2-y^2)}{(x^2+y^2)^3}, \tag{3}$$

which is continuous for $(x,y) \neq (0,0)$, since the denominator does not become 0 for such (x,y). At $(0,0)$ we have

$$(D_{21} f)(0,0) = \lim_{y\to 0} \frac{(D_1 f)(0,y) - (D_1 f)(0,0)}{y} = \lim_{y\to 0} \frac{-y-0}{y} = -1.$$

Let us consider what happens to $(D_{21}f)(x,y)$ when $(x,y) \to (0,0)$ along various rays, to show no overall limit exists. For fixed c, by (3),

$$(D_{21}f)(x,cx) = \frac{3-c^2}{1+c^2} - \frac{8c^2}{(1+c^2)^2} - \frac{(1+c^2)(2-6c^2) - 8c^2(1-c^2)}{(1+c^2)^3},$$

which we call $g(c)$. Since $g(c)$ is not a constant function of c (for example, $g(0) = 1, g(1) = -2$), the value $\lim_{x\to 0}(D_{21}f)(x,cx) = g(c)$ varies with c so indeed $\lim_{(x,y)\to(0,0)}(D_{21}f)(x,y)$ doesn't exist.

By (2), $(D_{12}f)(x,y) = -(D_{21}f)(y,x)$, so $(D_{12}f)(0,0) = 1$ and $\lim_{(x,y)\to(0,0)}(D_{12}f)(x,y)$ doesn't exist.

(I)(a) Apologies–this is not correct as stated, at least for the conorm; to make it true we need equal dimensions, $m = n$.

If $y \in T(U), y \neq 0$, then $y = Tx$ for some $x \in U$ with $x \neq 0$, so $|y| = |Tx| \leq |Tx|/|x| \leq ||T||$, meaning $y \in B(0, ||T||)$. Thus $T(U) \subset \overline{B}(0, ||T||)$. On the other hand, if $r < ||T||$, then there exists $x \neq 0$ with $|Tx|/|x| > r$, so for $z = x/|x|$ we have $z \in U$ but $Tz \notin \overline{B}(0,r)$. This shows $T(U) \not\subset \overline{B}(0,r)$, meaning $\overline{B}(0,||T||)$ is the smallest ball containing $T(U)$.

For the conorm, since $\mathfrak{m}(T) > 0$ we cannot have $x \neq 0, Tx = 0$. Therefore T is one-to-one, and thus onto (since $m = n$.) Suppose $y \in \overline{B}(0, \mathfrak{m}(T)), y \neq 0$. Since T is onto, we have $y = Tz$ for some $z \in \mathbb{R}^n, z \neq 0$. Then

$$\mathfrak{m}(T) \geq |y| = |Tz| \geq \mathfrak{m}(T)|z|,$$

where the last inequality is from the definition of $\mathfrak{m}(T)$, so $|z| \leq 1$, that is, $z \in U$, meaning $y \in T(U)$. This shows $\overline{B}(0, \mathfrak{m}(T)) \subset T(U)$. On the other hand, if $r > \mathfrak{m}(T)$, then by definition of infimum there exists $x \neq 0$ with $|Tx|/|x| < r$. This means we can find $c > 1$ with $c|Tx|/|x| < r$. Let $w = cx/|x|$, so $|w| = c > 1$, that is, $w \notin U$. Since T is one-to-one, this means $Tw \notin T(U)$. But we have $|Tw| = c|Tx|/|x| < r$, so $Tw \in \overline{B}(0,r)$. This shows $\overline{B}(0,r) \not\subset T(U)$. It follows that $\mathfrak{m}(T)$ is the radius of the largest ball contained in $T(U)$.

(b) We prove the contrapositive: if T is not invertible then $Tx = 0$ for some $x \neq 0$, so $|Tx|/|x| = 0$, which shows that $\mathfrak{m}(T) = 0$.

(II)(a) Given $\epsilon > 0$ we have

$$|y - x| < \frac{\epsilon}{||T||} \quad \text{implies} \quad |Ty - Tx| = |T(y-x)| \leq ||T||\ |y-x| < \epsilon.$$

Thus T is uniformly continuous.

(b) Let $f_n(x) = \frac{1}{n}\sin nx,\ x \in (a,b)$. Then $f_n \to 0$ uniformly and $T(0) = 0$, but we claim that $T(f_n) \not\to T(0) = 0$, which means T is not continuous. In fact $(Tf_n)(x) = f_n'(x) = \cos nx$, and for sufficiently large n there exist points of form $2\pi k/n$ in (a,b) with k an integer (such points exists provided $2\pi/n < b-a$.) At these points, $f_n'(2\pi k/n) = \cos 2\pi k = 1$. This shows $T(f_n)$ does not converge to 0 in $C(a,b)$, i.e. it does not converge uniformly to 0. Thus as claimed, T is not continuous.

But T is linear since $(cf+g)' = cf' + g'$.