MATH 425b ASSIGNMENT 3 SOLUTIONS
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Chapter 7

(20) Let $\epsilon > 0$. There exists a polynomial with $||P - f|| < \epsilon$ (sup norm), say $P(x) = \sum_{n=0}^{N} c_n x^n$. f is bounded since it is continuous on the compact set $[0,1]$, so there exists M such that $|f(x)| \leq M$ for all x. Therefore

$$\int_0^1 f(x)P(x)\ dx = \sum_{n=0}^{N} c_n \int_0^1 f(x)x^n\ dx = 0$$

and

$$\begin{aligned} 0 \leq \int_0^1 f(x)^2\ dx &= \left| \int_0^1 f(x)^2\ dx - \int_0^1 f(x)P(x)\ dx \right| \\ &= \left| \int_0^1 f(x)(f(x) - P(x))\ dx \right| \\ &\leq \int_0^1 |f(x)||(f(x) - P(x)|\ dx \\ &\leq \int_0^1 M\epsilon\ dx \\ &= M\epsilon. \end{aligned}$$

Since ϵ is arbitrary, this shows $\int_0^1 f(x)^2\ dx = 0$. By Exercise 2 of chapter 6, this means $f(x)^2 = 0$ for all x, so $f(x) = 0$ for all x.

(21) The constant function $f(e^{i\theta}) \equiv 1$ for all θ is in $\mathcal{A}$, and vanishes nowhere, so $\mathcal{A}$ vanishes at no point of K. The identity function $f(e^{i\theta}) = e^{i\theta}$ is in $\mathcal{A}$, and is one-to-one, so $\mathcal{A}$ separates points.

To prove Rudin's hint, for any function $f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta}$ in $\mathcal{A}$ we have

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\ d\theta = \sum_{n=0}^{N} c_n \int_0^{2\pi} e^{i(n+1)\theta}\ d\theta = 0. \tag{1}$$

For $f \in \overline{\mathcal{A}}$ there exists a sequence $\{f_n\} \subset \mathcal{A}$ with $f_n \to f$ uniformly. Hence applying (3) to

f_n,

$$\begin{aligned} \left| \int_0^{2\pi} f(e^{i\theta}) e^{i\theta}\, d\theta \right| &= \left| \int_0^{2\pi} (f(e^{i\theta}) - f_n(e^{i\theta})) e^{i\theta}\, d\theta \right| \\ &\le \int_0^{2\pi} |f(e^{i\theta}) - f_n(e^{i\theta})|\, |e^{i\theta}|\, d\theta \\ &\le 2\pi ||f -_n || \\ &\to 0 \quad \text{as } n \to \infty, \end{aligned} \tag{2}$$

so we must have $\int_0^{2\pi} f(e^{i\theta}) e^{i\theta}\, d\theta = 0$, for all $f \in \overline{\mathcal{A}}$. But for the particular choice $f(e^{i\theta}) = e^{-i\theta}$ we have $\int_0^{2\pi} f(e^{i\theta}) e^{i\theta}\, d\theta = 2\pi$, so $f \notin \overline{\mathcal{A}}$, though f is continuous on K.

Chapter 8

(1) We claim that for each n there is a polynomial P_n such that

$$f^{(n)}(x) = \begin{cases} P_n\left(\frac{1}{x}\right) e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

To prove this, note it is true for $n = 0$, with $P_n \equiv 1$. Suppose it is true for some n. The for $x \neq 0$,

$$\begin{aligned} f^{(n+1)}(x) &= P_n\left(\frac{1}{x}\right) \cdot \frac{2}{x^3} e^{-1/x^2} - \frac{1}{x^2} P_n'\left(\frac{1}{x}\right) e^{-1/x^2} \\ &= \left(\frac{2}{x^3} P_n\left(\frac{1}{x}\right) - \frac{1}{x^2} P_n'\left(\frac{1}{x}\right) \right) e^{-1/x^2}. \end{aligned} \tag{3}$$

Letting $P_{n+1}(x)$ be the coefficient of e^{-1/x^2} on the right side of (3), we see that since P_n and P_n' are both polynomials, so is P_{n+1}. Thus the claim follows for $x \neq 0$ by induction.

Given $x > 0$, by the Mean Value Theorem there exists $\xi \in (0, x)$ such that

$$\frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = f^{(n+1)}(\xi). \tag{4}$$

The same is true for $x < 0$, with $\xi \in (x, 0)$. Now $u^k e^{-u^2} \to 0$ as $u \to \infty$, for all $k \ge 0$, so $P_{n+1}(u) e^{-u^2} \to 0$ as $u \to \infty$, so $P_{n+1}\left(\frac{1}{x}\right) e^{-1/x^2} \to 0$ as $x \to 0$. Therefore $f^{(n+1)}(\xi) \to 0$ as $\xi \to 0$ (therefore also as $x \to 0$, since $\xi \in (0, x)$), so by (4) we have $f^{(n+1)}(0) = 0$. This completes the proof of the claim for $n + 1$, so it holds for all n by induction.

(4)(a) Let $f(x) = b^x = e^{(\log b)x}$, so $f'(x) = (\log b)e^{(\log b)x}$. Then

$$\lim_{x\to 0} \frac{b^x - 1}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = \log b.$$

(b) Use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{1} = 1.$$

(c) Use (b):

$$\lim_{x\to 0} (1+x)^{1/x} = \lim_{x\to 0} e^{\frac{\log(1+x)}{x}} = e^1 = e.$$

(d) By (c), $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{n/x} = e$. Since y^x is a continuous function of y, this shows

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n\to\infty} \left(\left(1 + \frac{x}{n}\right)^{n/x}\right)^x = e^x.$$

(5)(a) Let $g(x) = \frac{\log(1+x)}{x}$ for $x \neq 0$, so $(1+x)^{1/x} = e^{g(x)}$, and by 4b, $g(x) \to 1$ as $x \to 0$. Hence defining $g(0) = 1$ makes g continuous at 0. Using L'Hospital's Rule,

$$\begin{aligned}
\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} &= \lim_{x\to 0} \frac{\log(1+x) - x}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{1}{1+x} - 1}{2x} \\
&= \lim_{x\to 0} -\frac{(1+x^2)^{-2}}{2} \\
&= -\frac{1}{2},
\end{aligned}$$

so g is also differentiable at 0. Thus we can use the chain rule:

$$\begin{aligned}
\lim_{x\to 0} \frac{e - (1+x)^{1/x}}{x} &= \lim_{x\to 0} \frac{e^{g(0)} - e^{g(x)}}{x - 0} \\
&= -\frac{d}{dx} e^{g(x)} \,\Big|_{x=0} \\
&= -g'(0)e^{g(0)} \\
&= \frac{e}{2}.
\end{aligned}$$

(b) Let $f(x) = (e^x - 1)/x$, so we can express the quantity for which we want the limit as

$$f\left(\frac{\log n}{n}\right) = \frac{n}{\log n}(n^{1/n} - 1).$$

Letting $g(x) = e^x$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = g'(0) = 1,$$

so the desired limit is

$$\lim_{n\to\infty} f\left(\frac{\log n}{n}\right) = 1.$$

(6)(a) Taking $x = y = 0$ shows $f(0)^2 = f(0)$ so $f(0) = 0$ or 1 for all x. But $f(x) = f(x+0) = f(x)f(0)$ so if $f(0) = 0$ then $f(x)$ would be 0 for all x. Therefore $f(0) = 1$.

Let $g(x) = \log f(x)$, so $g(0) = 0$ and $g(x + y) = g(x) + g(y)$. Then

$$g'(x) = \lim_{h\to 0} \frac{g(x + h) - g(x)}{h} = \lim_{h\to 0} \frac{g(h)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0) \quad \text{for all } x.$$

Letting $c = g'(0)$, this shows that $g(x) = cx + c'$ for some c'. Since $g(0) = 0$ we must have $c' = 0$, so $g(x) = cx$, which means $f(x) = e^{cx}$.

(b) Rudin is a bit unclear—we still assume f is not 0, we just replace differentiability with continuity.

Since $g(x + y) = g(x) + g(y)$, taking $x = y$ shows $g(2x) = 2g(x)$, and then by easy induction on m,

$$g(mx) = g((m - 1)x + x) = g((m - 1)x) + g(x) = (m - 1)g(x) + g(x) = mg(x),$$

for all m and x. Hence also

$$g(x) = g\left(n \cdot \frac{x}{n}\right) = ng\left(\frac{x}{n}\right)$$

for all n, x, so $g(\frac{x}{n}) = \frac{1}{n}g(x)$. Therefore for all m, n,

$$g\left(\frac{m}{n}\right) = g\left(m \cdot \frac{1}{n}\right) = mg\left(\frac{1}{n}\right) = mg\left(\frac{1}{n} \cdot 1\right) = \frac{m}{n}g(1).$$

Letting $a = g(1)$ we thus have $g(x) = ax$ for all rational x. Since g is continuous, for irrational x we can take a sequence of rationals $x_k \to x$ and

$$g(x) = \lim_k g(x_k) = \lim_k ax_k = ax.$$

Thus $f(x) = e^{ax}$.

(A) ((a) $\implies$ (b)) Suppose $\sum_{n=0}^\infty a_n$ converges. Then the radius of convergence is at least 1, so f is defined at least on $[0, 1]$. For $x \in [0, 1]$ we have

$$|f(x) - \sum_{n=1}^{N} a_n x^n| \le \sum_{n=N+1}^{\infty} |a_n||x|^n \le \sum_{n=N+1}^{\infty} |a_n|.$$

The last sum does not depend on x, and approaches 0 as $N \to \infty$. Thus the series converges uniformly to $f(x)$ on $[0, 1]$.

((b) $\implies$ (c)) Suppose $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$. Then the limit $f(x)$ is a continuous function, so f is bounded on $[0, 1]$, hence also on $[0, 1)$.

((c) $\implies$ (a)) Suppose $\sum_{n=0}^{\infty} a_n = \infty$. Given $M > 0$ there exists N such that $\sum_{n=0}^{N} a_n > M$. Then for x sufficiently close to 1 we have $f(x) \geq \sum_{n=0}^{N} a_n x^n > M$. This shows that f is unbounded on $[0, 1]$.

(B) Suppose $g(x) \neq g(y)$ for some $x < y$. Let $\epsilon, \delta > 0$; we will show that δ does not "work" for this ϵ for some multiple of g. Essentially this is because if you multiply g by a big enough constant c, the increments of g become very large. Specifically, let $x = x_0 < x_1 < ... < x_n = y$ with $x_i - x_{i-1} < \delta$ for all $1 \leq i \leq n$. If δ "works" for some positive multiple cg of g, then

$$c|g(y) - g(x)| = |cg(y) - cg(x)| < c\left|\sum_{i=1}^{n}(g(x_i) - g(x_{i-1}))\right| \leq \sum_{i=1}^{n}|cg(x_i) - cg(x_{i-1})| \leq n\epsilon.$$

But for large c this is false, because $c|g(y) - g(x)| > n\epsilon$. In other words, δ does not "work" for all multiples cg of g. Thus $\mathcal{A}$ is not equicontinuous.

(C)(a) Since f is never 0, $\mathcal{A}_1$ vanishes at no point of $[0, 1]$. If $(x_1, y_1) \neq (x_2, y_2)$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ then $g(x_1, y_1) \neq g(x_2, y_2)$. If $y_1 \neq Y_2$ then $f(x_1, y_1) \neq g(x_2, y_2)$. This shows that $\mathcal{A}_1$ separates points. By the Stone-Weierstrass Theorem, the uniform closure of $\mathcal{A}_1$ is all of $C([0, 1]^2)$, so in particular it includes h.

(b) Every polynomial of form $c + (x - \frac{1}{2})^2 R(x)$, with R a polynomial and c a constant, is in $\mathcal{A}_2$. In particular the strictly increasing function $(x - \frac{1}{2})^3 \in \mathcal{A}_2$, which shows that $\mathcal{A}_2$ separates points. Taking $c > 0$ and $R \equiv 1$ we see that $\mathcal{A}_2$ vanishes at no point. By the Stone-Weierstrass Theorem, $\mathcal{A}_2$ is dense in $C[0, 1]$.