MATH 425b ASSIGNMENT 3
SPRING 2009
Prof. Alexander
Due Wednesday February 18.

Rudin Chapter 7 #20, 21; Chapter 8 #1, 4, 5ab, 6 and:

(A) Let $f(x) = \sum_{n=-}^{\infty} a_n x^n$ be a power series. An example was given in lecture to show that it's possible for such a series to have radius of convergence 1, with $f(x)$ staying bounded as $x \to 1$, yet the series not converging for $x = 1$. This problem shows that is not possible if the coefficients a_n are nonnegative. Suppose all $a_n \geq 0$, and show that the following are equivalent:

(a) $f(1) = \sum_{n=1}^{\infty} a_n$ converges;
(b) The power series converges uniformly on $[0, 1]$;
(c) f is bounded on $[0, 1)$.

(B) Show that if $\mathcal{A}$ is an algebra of real-valued functions on $[a, b]$ and $\mathcal{A}$ contains a nonconstant function, then $\mathcal{A}$ is not equicontinuous.

(C) (a) Let $\mathcal{A}_1$ be an algebra of real-valued functions on $[0, 1]^2$ which contains the functions $f(x, y) = e^y$ and $g(x, y) = 1/(x + 2)$. Does the uniform closure of $\mathcal{A}_2$ necessarily include the function $h(x, y) = \sin xy$?

(b) Let $\mathcal{A}_2$ be the algebra consisting of all polynomials f on $[0, 1]$ satisfying $f'(\frac{1}{2}) = 0$. Is $\mathcal{A}_2$ dense in $C[0, 1]$?

HINTS:

(20) You may find Exercise 2 in Chapter 6 useful–you may take it as given.

(21) Notice that in the Stone-Weierstrass Theorem (7.32), the functions are real-valued, so that theorem is not violated by this problem!

You need to find a continuous function f on K with $\int_0^{2\pi} f(e^{i\theta})e^{i\theta}\, d\theta \neq 0$. What's a simple sort of function that has a nonzero integral?

(1) Prove that for all n, $f^{(n)}(x)$ has the form $P_n(\frac{1}{x})e^{-1/x^2}$ for $x \neq 0$, where P_n is a polynomial. You don't need to actually find P_n. What proof method is suited to proving such a statement "for all n"?

(4) Try to do (a), (c), (d) without l'Hopital's Rule:

(a) Find a way to use Theorem 8.6b.
(c) Use (b).
(d) Use (c).

(5)(a) Write $(1+x)^{1/x}$ as $e^{g(x)}$ for some $g(x)$. Then

$$\frac{e-e^{g(x)}}{x}=\frac{e^1-e^{g(x)}}{1-g(x)}\frac{1-g(x)}{x}.$$

(b) Turn the expression into a difference quotient.

(6)(a) Plugging in the right x, y gives you $f(0)$. Then show $\log f(x)$ has a constant derivative.

(b) Four steps: (i) Show $g(mx) = mg(x)$ for positive integers m. (ii) Prove an analog of (a) for $g(x/n)$ for positive integers n. (iii) Use (i), (ii) to relate $g(m/n)$ to $g(1)$. (iv) Use (iii) and continuity of g to complete the proof.

(A) Show (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c), and "not (a)" $\Longrightarrow$ "not (c)."

(B) Consider scalar multiples of some nonconstant function $g \in \mathcal{A}$.

(C)(b) It may be useful to find a particular degree-2 polynomial $Q(x)$ such that if you multiply any polynomial by $Q(x)$ you get a polynomial in $\mathcal{A}_2$.