MATH 425b FINAL EXAM
May 11, 2009
Prof. Alexander

Problem	Points	Score
1	45	
2	40	
3	35	
4	30	
Total	150	

Name: ______________________________

USC ID: ______________________________

Signature: ______________________________

Notes:

SHOW ALL WORK.

Use the backs of the sheets if you need more space.

If you want full credit, you need your logic to be clearly explained. Also, if you leave a gap in the logic, you can get more partial credit if you recognize the gap, i.e. say "I need to show.....here but I'm not sure how," rather than just jump over that step.

If you can't do part (a) of a problem, you can assume it to do part (b).

(1) These are 3 separate problems about forms; the assumptions for each problem don't apply to the other problems.

(a) Let $\omega = f(x)\ dx_1 \wedge dx_2 + g(x)\ dx_3 \wedge dx_4$ be a 2-form in some $\Omega \subset \mathbb{R}^n$, with f, g continuous. Under what conditions on f, g does $\omega \wedge \omega \neq 0$?

(b) Suppose $\omega = f(x)\ dx_I + g(x)\ dx_J$ is a k-form, and k is odd. Show that $\omega \wedge \omega = 0$. HINT: (53) in Chapter 10. (The result is actually true for general k-forms—I used one here with two terms just to keep things simpler.)

(c) Suppose ω is an exact k-form in some open $E \subset \mathbb{R}^n$. Show that $\omega \wedge d\beta$ is exact for every form β in E. HINT: This is very short if you apply the right theorem.

(2) Let $f^{(k)}$ and f be functions in L^2 having Fourier coefficients $\{c_n^{(k)}, n \in \mathbb{Z}\}$ and $\{c_n, n \in \mathbb{Z}\}$ respectively.

(a) Show that $f^{(k)} \to f$ in L^2 (that is, $||f^{(k)} - f||_2 \to 0$) if and only if $\sum_{n\in\mathbb{Z}} |c_n^{(k)} - c_n|^2 \to 0$ as $k \to \infty$.

(b) Suppose that for each $n \in \mathbb{Z}$, we have $c_n^{(k)} \to c_n$ as $k \to \infty$, and suppose there is a sequence $\{b_n, n \in \mathbb{Z}\}$ such that $|c_n^{(k)}| \leq b_n$ for all k, and $\sum_{n\in\mathbb{Z}} b_n^2 < \infty$. Show that $f^{(k)} \to f$ in L^2. HINT: Use (a). Decompose $\sum_{n\in\mathbb{Z}}$ into $\sum_{n\in[-N,N]} + \sum_{n\notin[-N,N]}$, for some appropriate N. One of these two sums can be made small using $\sum_{n\in\mathbb{Z}} b_n^2 < \infty$.

(3) Let K be a compact metric space and for $c > 0$ define the sets of Lipschitz functions

$$\mathrm{Lip}_c(K) = \{f : K \to \mathbb{R} : |f(x) - f(y)| \le c d(x, y) \text{ for all } x, y \in K\}, \quad \mathrm{Lip}(K) = \cup_{c=1}^{\infty} \mathrm{Lip}_c(K).$$

An example of a nonconstant Lipschitz function in a general metric space is $f(x) = d(x, x_0)$ for some fixed x_0. As usual, $C(K)$ denotes the set of all continuous functions on K, endowed with the uniform (sup) metric.

(a) Show that $\mathrm{Lip}(K)$ is an algebra, that is, if $f, g \in \mathrm{Lip}(K)$ and $a \in \mathbb{R}$ then $af, f + g$ and fg are in $\mathrm{Lip}(K)$.

(b) Show that $F_{c,M} = \{f \in \mathrm{Lip}_c(K) : ||f||_\infty \le M\}$ is a compact subset of $C(K)$, for each $c, M > 0$. HINT: We proved a criterion for a subset of $C(K)$ to be compact.

(4)(a) Consider a distribution of mass described by the mass density $\rho(x), x \in \mathbb{R}^3$, which we assume is continuous. This gives rise to a gravitational field $F = (F_1, F_2, F_3)$, which is $\mathcal{C}'$. Let Ω be a closed subset of $\mathbb{R}^3$ with positively oriented boundary $\partial\Omega$. Let $m(\Omega)$ be the total amount of mass in Ω, and let G be the gravitational constant. A physical law states that for all such Ω, the total flux across the boundary satisfies

$$\iint_{\partial\Omega} F \cdot \mathbf{n}\, dA = -4\pi G m(\Omega).$$

Show that, as a consequence of this law, we have

$$\text{div } F = -4\pi G\rho.$$

HINT: What can you conclude if two functions have the same integral over every region?