Notes:
The exam is closed book, no calculator, but you are allowed one 8 1/2 × 11 handwritten sheet of notes (two sides.) It takes place 8–10 a.m. Monday May 8, in the lecture room (SOS B46).

(1)(a) For $A = \begin{bmatrix} 1 & 1 & -3 & 2 \\ -1 & -2 & 3 & 1 \\ 3 & 7 & -9 & -6 \end{bmatrix}$, find all solutions of $Ax = 0$.

(b) Find a basis for the solution set you found in (a).

(c) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and let $v_2, v_3$ be the basis vectors you found in (b). Find a vector which is not in the linear span of $v_1, v_2, v_3$.

(d) What is the dimension of the range of $A$? Say how you know.

(2)(a) Find the matrix $A$ of the linear transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ which doubles the horizontal coordinate, that is, $T((a, b)) = (2a, b)$.

(b) The matrix for the linear transformation described by “rotate counterclockwise by 45°” is $B = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. Find the matrix of the linear transformation described by “double the horizontal coordinate, then rotate by 45°.”

(3)(a) Use the cofactor method (not other methods) to calculate the inverse of the matrix $A = \begin{bmatrix} -4 & 1 & -3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

(b) Are the rows of $A$ linearly independent? How do you know?

(4)(a) Can a non-square $m \times n$ matrix have an eigenvector? Explain.

(b) Suppose $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of a $2 \times 2$ matrix $A$ with eigenvalue 3, and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a generalized eigenvector: $(A - 3I)e_2 = e_1$. What is the matrix $A$? HINT: Find the columns of $A$, using the information given.
(5) Let \( A = \begin{bmatrix} -2 & 8 \\ -2 & 6 \end{bmatrix} \).

(a) Solve the initial value problem \( x' = Ax \), \( x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).

(b) Does a solution exist to \( x' = Ax \), \( x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), \( x'(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \)? Say how you know.

(c) Is there a matrix \( S \) for which \( S^{-1}AS \) is a diagonal matrix \( D \)? If so, find \( S \) and \( D \). If not, say how you know.

(6) Let \( P(D) = (D^2 + 6D + 13)(D - 5)^2 \).

(a) Find the general solution of \( P(D)y = 0 \).

(b) Give the proper form for a particular solution to \( P(D)y = \cos t \). You need not calculate any coefficients, just write \( A_0, A_1, \) etc.

(c) Give the proper form for a particular solution to \( P(D)y = 9te^{3t} \). Again, you need not calculate any coefficients, just write \( A_0, A_1, \) etc.

(d) Find the general solution of \( P(D)y = 32e^{-3t} \cos 2t \). To avoid excessive computation, you may take the following as given:

\[
P(D)(te^{-3t} \cos 2t) = 32e^{-3t}(8 \cos 2t - 15 \sin 2t),
\]
\[
P(D)(te^{-3t} \sin 2t) = 32e^{-3t}(15 \cos 2t + 8 \sin 2t).
\]

HINT: It is better to use some method other than row reduction for the systems of equations that you need to solve.

(7) (a) Suppose \( y_1(t) = 4t^2 \sin t \) and \( y_2(t) = 5t^2 \cos t \) are solutions of \( y'' + p(t)y' + q(t)y = \sin t \). Find a solution \( y(t) \) for the corresponding homogeneous equation \( y'' + p(t)y' + q(t)y = 0 \).

(b) Let \( P(D) \) be a linear differential operator and let \( S = \{ (a_1, a_2) \in \mathbb{R}^2 : y(t) = a_1t^2 + a_2t^3 \text{ is a solution of } P(D)y = 0 \} \). Show that \( S \) is a subspace of \( \mathbb{R}^2 \). (Note that we do not assume \( t^2 \) and \( t^3 \) are solutions.)

(8) (a) Find the general solution of \( y'' + 8y' + 16y = 0 \).

(b) Find the general solution of \( y'' + 8y' + 16y = \sqrt{t} e^{-4t} \). HINT: “Undetermined coefficients” does not work well here—try variation of parameters.