A Numerical Algorithm for Solving Models With Incomplete Markets
Ayse Imrohoroglu, Selahattin Imrohoroglu and Douglas H. Joines
International Journal of High Performance Computing Applications 1993 7: 212
DOI: 10.1177/109434209300700304

The online version of this article can be found at:
http://hpc.sagepub.com/content/7/3/212

Published by:
SAGE
http://www.sagepublications.com

Additional services and information for International Journal of High Performance Computing Applications can be found at:

Email Alerts: http://hpc.sagepub.com/cgi/alerts
Subscriptions: http://hpc.sagepub.com/subscriptions
Reprints: http://www.sagepub.com/journalsReprints.nav
Permissions: http://www.sagepub.com/journalsPermissions.nav
Citations: http://hpc.sagepub.com/content/7/3/212.refs.html

>> Version of Record - Sep 1, 1993

What is This?
A NUMERICAL ALGORITHM FOR SOLVING MODELS WITH INCOMPLETE MARKETS

Ayşe İmrohoroğlu, Selahattin İmrohoroğlu, and Douglas H. Joines

DEPARTMENT OF FINANCE AND BUSINESS ECONOMICS
SCHOOL OF BUSINESS ADMINISTRATION
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CALIFORNIA 90089-1421

Summary

We describe an incomplete markets model and outline a numerical solution algorithm to compute its steady-state equilibrium. The model departs from the Arrow-Debreu world of complete contingent claims markets by assuming the presence of borrowing constraints. Solution methods that rely on approximations around some reference point or those that use the Euler equations in an interior solution are not suitable for the present model. Rather, we outline a numerical solution algorithm that discretizes the state space and the decision space and iterates on a numerical procedure that obtains the decision rules and distributions of agent types for all cohorts in an attempt to find the steady-state equilibrium of the model. The algorithm is implemented using the Fortran compilers on the Minnesota CRAY X-MP and San Diego CRAY Y-MP/8-864.

Introduction

The infinite-horizon representative agent framework has been the workhorse of macroeconomics for the last decade. In such environments, it is assumed that the Arrow-Debreu contingent claims markets are complete, and as a result the distribution of wealth becomes irrelevant to computation of the equilibrium laws of motion. This setup has obvious computational advantages since equilibrium aggregate variables can be found without keeping track of the distribution of wealth. This framework has been used to address many interesting questions related to the study of business cycles, monetary and fiscal policy, endogenous growth, and optimal taxation (see, among others, Kydland and Prescott, 1982; Hansen, 1985; Lucas, 1987, 1988, 1990; Lucas and Stokey, 1983; Rebelo, 1991; Jones and Manuelli, 1991; and Sargent 1987). This abstraction, however, assumes full and costless insurance of idiosyncratic risk through the existence of complete markets. It is then natural to ask how the results obtained from the representative agent world would change when one departs from the complete markets setup.

One way to limit insurance arrangements is to follow Scheinkman and Weiss (1986) and preclude perfect insurance exogenously. Such an approach is followed by İmrohoroğlu (1989), who examines the welfare cost of business cycles in this setup. In her infinite-horizon model, individuals face an uncertain income because of the variability of the work option. They self-insure against future fluctuations in their earnings by saving during periods in which they are employed. When they are unemployed, they dissave, but they are restricted to hold nonnegative asset holdings. The possibility that optimal consumer behavior may at times occur at a corner solution implies that standard numerical solution algorithms like the linear-quadratic method or Euler-equation-based methods like those of Judd (1991), Coleman (1990), and Den Haan and Marcet (1990) are not suitable for this model. The method used in İmrohoroğlu (1989) begins by writing the individuals' optimization problem as a finite-state, discounted, dynamic program. Then, the state and control spaces are discretized, and a standard successive approximation method is used to obtain the decision rules, time-
invariant distribution of agent types, and equilibrium values for the aggregate variables.

The individuals in İmrohoroğlu's (1989) model are infinitely lived. For analyzing some economic issues, however, a life-cycle feature is essential. For example, to study the impact of an unfunded social security system on the U.S. capital stock and individual welfare, one has to specify a model populated with finite-lived individuals. This article provides the details of such a model economy and a numerical algorithm to solve the model. The main features of this overlapping generations model are borrowing constraints, lifetime uncertainty, and incomplete insurance of idiosyncratic income risk. This model, like the infinite-horizon one, cannot be solved by linear-quadratic or Euler-equation-based methods.

The next section summarizes the infinite-horizon model with incomplete markets and discusses the solution method for that model. We then present the overlapping generations version of the model, discuss the solution method in detail, and present some illustrative results obtained by applying the model to an analysis of an unfunded social security system.

Infinite-Horizon, Finite-State, Dynamic Programming

The representative agent framework assumes the existence of a full set of Arrow-Debreu contingent claims markets and hence full and costless insurance against idiosyncratic income risk. It is natural to ask how the quantitative results obtained from the representative agent world would change when one departs from the complete markets setup. This section describes the infinite-horizon, incomplete markets environment of İmrohoroğlu (1989) and summarizes the results of the experiment conducted therein.

The economy consists of a continuum of infinitely lived individuals who are ex ante identical but ex post heterogeneous with respect to their employment opportunities. The resulting variability in income cannot be eliminated through insurance, because markets are assumed to be incomplete. Consequently, the variation in employment opportunities induces heterogeneity in
consumption and asset holdings. The departure from the Arrow-Debreu setup is implemented by restricting asset holdings to be nonnegative in each time period.

Individuals are endowed with an indivisible unit of time in each period that can be used for leisure or work effort. They face an idiosyncratic employment opportunity that follows a two-state, first-order Markov process. If \( s = e \), then the individual supplies the time endowment elastically and produces \( y \) units of output. (It is relatively easy to incorporate a nontrivial labor-leisure choice. Hansen and Imrohoroglu [1992] make labor endogenous and study the impact of moral hazard on the optimality of unemployment insurance benefits. Diaz-Gimenez, et al. [1992] investigate the welfare effects on the economy in a heterogeneous-agent economy with an explicit banking sector.) If \( s = u \), then the individual is unemployed and gets \( 6y \) units of output where \( 0 \in (0,1) \). The transition matrix for \( s \) is given by the 2 \( \times \) 2 matrix \( \Pi = [\pi_{ij}] \) where \( \pi_{ij} = \Pr (s_{t+1} = j | s_t = i) \). (Here, we will describe the version of the model without aggregate shocks. Including aggregate uncertainty in this version of the model is straightforward. The important difference is in the description of an individual's employment status, which depends not only on the idiosyncratic risk but also on aggregate fluctuations. However, it should be mentioned that combining aggregate uncertainty and endogenous factor prices produces an unsolvable problem because in this case the measure of agent types itself becomes a state variable.)

Let \( a_t \) denote the asset holdings of an individual at the beginning of period \( t \), and \( r \) denote the net rate of return on \( a_t \). Because asset holdings must be nonnegative in each time period, individuals have to save in order to self-insure against fluctuations in their endowments. The budget constraint faced by the agents is given by

\[
a_t \geq 0, \quad a_{t+1} = \begin{cases} (1 + r)(a_t - c_t + y) & \text{if } s = e, \\ (1 + r)(a_t - c_t + \theta y) & \text{if } s = u. \end{cases}
\]

(1)

Individuals rank consumption sequences based on their expected discounted utility:

\[
E \sum_{t=0}^{\infty} \beta^t U(c_t),
\]

(2)

where \( \beta \in (0,1) \) is their subjective discount factor, and \( c_t \) is their time-\( t \) consumption.

The utility maximization problem faced by an individual can be stated as a finite-state, discounted, dynamic program. Let \( a \) and \( s \) be the state variables, and \( a' \) and \( s' \) denote next period values for these variables, respectively. The optimality equation can be written as

\[
V(a,s) = \max_a \left\{ U(c) + \beta E V(a',s') \right\},
\]

(3)

subject to statement (1) and \( a, \geq 0 \), where \( V(a,s) \) is the value function and \( E \) is the expectations operator evaluated over the distribution of \( s \). After the budget constraint (1) is used to substitute out \( c \) in the above problem, the standard successive approximation method to obtain the value function \( V(a,s) \) and the decision rule \( a' = f(a,s) \) is to iterate on the following

\[
V_{k+1}(a,s) = \max_a \left\{ U(a,s,a') + \beta \sum_{s'} \Pi(s,s') V_k(a',s') \right\},
\]

(4)

starting from an initial value function \( V_0 \) where \( V_0(a,s) \) is the \( k \)th approximation to the value function \( V(a,s) \). Even in the representative agent framework, analytic solution to the above problem is known for very few and restricted classes of utility functions. In the present case, numerical methods must be used to obtain the solution to the individual's problem. (Since there is a positive probability that an individual's optimal policy is at a corner, numerical solution methods that rely on approximations around a steady state like the linear-quadratic method, or those that use the Euler equations of an interior solution like Judd [1991], Coleman [1990], or Den Haan and Marcet [1990] are not suitable for this class of Kuhn-Tucker problems.)

The solution method used in Imrohoroglu (1989) is to discretize the state and decision spaces and to obtain exact numerical solutions to the discrete economy. A grid of 301 points with increments of 0.027 in asset holdings is used. Given that the employment state has two realizations, the total number of points in the state space is 301 \( \times \) 2 = 602. The decision space also has 602 points. As a result, an individual's decision rule is a 301 \( \times \) 2 matrix whose elements are required to fall on the grid of 301 points in asset holdings. The value function is also a 301 \( \times \) 2 matrix. Imrohoroglu (1989) uses the standard method of successive approximation by taking
an initial value function $V_0(a,s)$ and using the update rule in Eq. (4) to obtain the decision rule at the end of the first iteration $f_1(a,s)$ and the value function at this iteration $V_1(a,s)$. Using Eq. (4) again yields $f_2(a,s)$ and $V_2(a,s)$. The convergence criterion used in İmrohoğlu (1989) is that two successive decision rule matrices be identical. Since two adjacent grid points are apart from each other by the increment 0.027, this is equivalent to using a convergence criterion of 0.027. Until this is satisfied, iterations continue. The sequence of decision rules in İmrohoğlu (1989) typically converge after about 120 iterations. Once the decision rules converge, the commodity market clearing condition is guaranteed to hold since the market clearing condition simply integrates both sides of the individuals' budget constraints, which are required to hold during the computation of the decision rules.

After the decision rules have been computed, the 602 x 602 equilibrium state transition matrix is checked for ergodicity. If ergodicity is satisfied, then one can utilize two related results. First, there exists a unique time-invariant measure of agent types $\lambda^*(a,s)$ which in this case is a 301 x 2 matrix whose typical element has two interpretations. It is the fraction of the population, at a given point in time, that resides in that particular employment state and asset-holding level. Alternatively, it is the fraction of time an individual is expected to spend in a particular employment state and asset-

1. For more details, the reader is referred to the original source.

There are several devices one can use to reduce the computational burden of the problem. One device is to evaluate the initial value function $V_0(a,s)$ at the deterministic steady state of the model in which the random variables are set equal to their unconditional means. This reduces the number of iterations by about 30%. Another device to reduce the CPU time is to tabulate the utility function for each point in the state space in the beginning of the code and to store this $2 \times 301 \times 301$ matrix in core memory and read off of the memory as requested by the code, instead of evaluating the utility function $2 \times 301 \times 301$ times at each of the possible 120 iterations. The interested reader can see Alvarez and Fitzgerald (1992) for a similar solution algorithm and its convergence properties. Sargent (1992) and Alvarez and Fitzgerald (1992) describe a policy iteration algorithm (or, equivalently, the Howard policy improvement algorithm), which is shown to provide considerable improvement in computational speed. Another device used by Alvarez and Fitzgerald (1992) is to use a linear interpolation rule to reduce the number of utility function evaluations.

"The convergence criterion used is that two successive decision rule matrices be identical. Until this is satisfied, iterations continue. Once the decision rules converge, the commodity market clearing condition is guaranteed to hold since the market clearing condition simply integrates both sides of the individuals' budget constraints, which are required to hold during the computation of the decision rules."
"For some economic issues, a life-cycle feature is essential. For example, in order to study the impact of an unfunded social security system on the U.S. capital stock and on individual welfare, one has to specify a model populated with finite-lived individuals. We describe such a model economy and present a numerical algorithm to solve the model."

Finite-Horizon, Finite-State, Dynamic Programming

For some economic issues, a life-cycle feature is essential. For example, in order to study the impact of an unfunded social security system on the U.S. capital stock and on individual welfare, one has to specify a model populated with finite-lived individuals (the literature on the effects of social security is very large; see Auerbach and Kotlikoff, 1987; Hubbard and Judd, 1987; and other references in İmrohoroğlu, İmrohoroğlu, and Joines, 1992). This section describes such a model economy and presents a numerical algorithm to solve the model. The main features of the overlapping generations model presented here are borrowing constraints, lifetime uncertainty, and incomplete insurance against idiosyncratic income risk.
DESCRIPTION OF MODEL ECONOMY

The economy is populated with a continuum of individuals with total measure one. After birth at age $j = 1$, individuals survive from age $j - 1$ to age $j$ with probability $\psi_j$. After age $j = J$, death is certain. There is an unconditional probability $\Pi_{J} = 1$ that an individual will survive to period $J$. However, a significant portion of the individuals will experience early death and leave accidental bequests. We assume that these bequests are taxed 100% by the government and then rebated back to the survivors (of all ages) in a lump-sum fashion.\(^2\)

At each date $t \geq 1$, a new generation is born with a population share given by the measure $\mu_j = 1, 2, \ldots, J$. (The lack of a time subscript on $\mu_j$ indicates our assumption that the population is stable.) The shares are given by $\mu_{j+1} = \psi_{j+1}/\rho \mu_j$, where $\rho$ is the population growth rate and $\Sigma_{j=1}^{J} \mu_j = 1$. Individuals maximize the expected, discounted lifetime utility

$$E \sum_{j=1}^{J} \beta^{j-1} \left[ \prod_{k=1}^{j} \psi_k \right] U(c_j),$$

where $c_j$ is consumption of an age-$j$ individual. (We will restrict attention to computing steady-state equilibria. Hence, there will be no time subscripts throughout the paper. It is computationally very demanding to calculate equilibria along a transition path or in general out of a steady state.) The utility function is assumed to take the form

$$U(c_j) = \frac{c_j^{1-\gamma}}{1-\gamma},$$

where $\gamma$ is the coefficient of risk aversion.

Each period, individuals who are below an exogenously given mandatory retirement age, $j^*$, face a stochastic employment opportunity. Let $s \in S = \{e, u\}$ denote the employment opportunities state and assume that it follows a two-state, first-order Markov process. If $s = e$, the agent is given the opportunity to work. If $s = u$, the agent is unemployed. The transition function for the employment opportunities state is given by the $2 \times 2$ matrix $\Pi(s',s) = [\pi_{ij}], i, j = e, u$, where $\pi_{ij} = \text{Prob}(s_{t+1} = j \mid s_t = i)$. Agents in this economy supply labor inelastically whenever they are given an opportunity to work. (Again, it is straightforward to amend the solution method to accommodate an endogenous labor-leisure choice.) Let $w$ and $r$ denote the wage rate (in terms of the consumption good) and the rate of return on asset holdings, respectively. Let $\epsilon_j$ denote the efficiency index (the number of units of work effort into which one unit of time can be turned) of an age-$j$ agent. Denote work effort and end-of-period asset holdings at age $j$ by $n_j$ and $a_j$, respectively. Agents in this economy are not allowed to borrow. This borrowing constraint can be stated as

$$a_j \geq 0, \forall j.$$

Before the mandatory retirement age of $j^*$, an individual receives the realization of the idiosyncratic employment shock. If $s = e$, then $n_j = \hat{h}$ and the individual receives $w_j = w\epsilon_j h$, where $w$ is the wage when employed, $\epsilon_j$ is the efficiency index of individuals of age $j$, and $\hat{h}$ is the number of hours worked by an age-$j$ individual. If $s = u$, then $n_j = 0$, and the individual receives unemployment insurance benefits in the amount $u_j = \xi u \hat{h}$, where $\xi$ is the replacement ratio. After the mandatory retirement age of $j^*$, the disposable income of a retiree is equal to the social security benefit, $b_j$. These benefits are calculated to be a fraction, $\theta$, of some base income, which we take as the average lifetime employed income. That is,

$$b_j = \begin{cases} 0, & \text{for } j = 1, 2, \ldots, j^* - 1; \\ \sum_{j'=1}^{j^*-1} w_{j'}, & \text{for } j = j^*, j^* + 1, \ldots, J, \end{cases}$$

\(^2\) There are alternative redistribution schemes. For example, these unintended bequests can be given to the newly born in a lump-sum fashion, they can be disposed without providing any use to anyone. Still another scheme would distribute these accidental bequests to the survivors of the same cohort in proportion to their wealth. This last scheme is equivalent to a mandatory annuity contract that allows the individuals to insure themselves against lifetime uncertainty. The presence of such an annuity market may change the quantitative results significantly, as in Imrohoroglu (1992).
The only role of government in this economy is to administer the unemployment insurance and social security programs. Given unemployment insurance and social security benefits, the government chooses the unemployment insurance and the social security tax rates so that its budget is balanced. Under these assumptions, the disposable income of an individual is given by

\[ q_j = \begin{cases} 
(1 - \tau_u - \tau_s)w_j^s & \text{for } j = 1, 2, \ldots, j^* - 1, \text{if } s = u, \\
q_j & \text{for } j = 1, 2, \ldots, j^* - 1, \text{if } s = s, \\
b_j & \text{for } j = j^*, j^* + 1, \ldots, J. 
\end{cases} \]

(9)

Then, the budget constraint facing an individual can be written as

\[ a_j = (1 + r)a_{j-1} + q_j - c_j + T, \text{ } y_0 \text{ given}, \]

where \( q_j \) is the disposable income of an age-\( j \) individual and \( T \) denotes the lump-sum distribution of unintended bequests.

The particular social security arrangements in place are described by the pair \((\theta, \tau_s)\), which represent the replacement rate and the payroll tax rate for social security. The unemployment insurance replacement ratio and the associated tax rate are also part of government’s policy specification. The only condition for choosing the policy instruments is that both the social security and the unemployment insurance systems be self-financing.

The production technology of the economy is given by a constant returns to scale Cobb-Douglas function

\[ Q = f(K, N) = BK^{1-\alpha}N^\alpha, \]

(11)

where \( B > 0, \alpha \in (0,1) \) is labor’s share of output, and \( K \) and \( N \) are aggregate capital and labor inputs, respectively. The aggregate capital stock is assumed to depreciate at the rate \( \delta \). The profit-maximizing behavior of the firm gives rise to first-order conditions that determine the net real return to capital and the real wage

\[ r = (1 - \alpha)B \left( \frac{K}{N} \right)^{-\alpha - \delta}, \text{ } w = \alpha B \left( \frac{K}{N} \right)^{1-\alpha}. \]

(12)

In order to describe the individual’s utility maximization problem as a finite-state, discounted, dynamic program, we define the constraint set as follows. Let \( D = \{d_1, d_2, \ldots, d_m\} \) denote the discrete grid of points on which asset holdings will be required to fall. For any beginning-of-period asset holding and employment status \((a, s) \in D \times S\), define the constraint set of an age-\( j \) agent \( \Omega_j(a, s) \in R_+ \) as all pairs \((c_j, a_j)\) such that the following are satisfied for \( j = 1, 2, \ldots, J\):

\[ a_j = (1 + r)a_{j-1} + q_j - c_j + T, \text{ } y_0 \text{ given}, \]

\[ c_j \geq 0. \]

(13)

Let \( V_j(a, s) \) be the value of the objective function of an age-\( j \) agent with beginning-of-period asset holdings and employment status \((a, s)\). \( V_j(a, s) \) is defined as the solution to the dynamic program

\[ V_j(a, s) = \max_{(c, a') \in \Omega_j(a, s)} \{ U(c) + \beta \psi_{j+1} V_{j+1}(a', s') \}. \]

(14)

COMPETITIVE EQUILIBRIUM

We can now define an equilibrium for the present model economy.

DEFINITION: An equilibrium for a given set of policy arrangements \( \{\theta, \xi, \tau_u, \tau_s\} \) is a collection of value functions \( V_j(a, s) \), individual policy rules \( C_j : D \times S \to R_+, A_j : D \times S \to D \), age-dependent (but time-invariant) measures of agent types \( \lambda_j(a, s) \) for each age \( j = 1, 2, \ldots, J \), relative prices of labor and capital \( \{w, r\} \), and a lump-sum transfer \( T^* \) such that:

i. Aggregate variables are computed from individual behavior:

\[ K = \sum_j \sum_{a} \sum_{s} \mu_j \lambda_j(a, s) A_{j-1} \text{ and } N = \sum_{j=1}^{J-1} \sum_{a} \mu_j \lambda_j(a, s) = r\epsilon h. \]

(15)

ii. The relative prices \( \{w, r\} \) solve the firm’s profit maximization problem by satisfying Eq. (12).

iii. Given relative prices \( \{w, r\} \), government policy \( \{\theta, \xi, \tau_u, \tau_s\} \), and a lump-sum transfer \( T^* \), the individual policy rules \( C_j(a, s), A_j(a, s) \) solve the individuals’ dynamic program Eq. (14).

iv. The commodity market clears,

\[ \sum_j \sum_{a} \sum_{s} \mu_j \lambda_j(a, s) [C_j(a, s) + A_j(a, s)] = f(K, N) + (1 - \delta) \sum_j \sum_{a} \sum_{s} \mu_j \lambda_j(a, s) A_{j-1}(a, s). \]

(16)
where the initial wealth distribution of agents, \(A_0\), is taken as given.

v. The collection of age-dependent, time-invariant measures \(\lambda_j(a,s)\) for \(j = 1, 2, \ldots, J\), satisfies
\[
\lambda_j(a',s') = \sum_{s} \sum_{a' = A_j(a,s)} \Pi(s',s) \lambda_{j-1}(a,s),
\]  
(17)
where the initial measure of agents at birth, \(\lambda_j\), is taken as given.

vi. The social security system is self-financing:
\[
\tau_u = \frac{\sum_{j=1}^{J} \sum_{a} \mu_j \lambda_j(a,s) b_j}{\sum_{j=1}^{J} \sum_{a} \mu_j \lambda_j(a,s) e_j + \sum_{j=1}^{J} \sum_{a} \mu_j \lambda_j(a,s) \epsilon_j}.
\]  
(18)

vii. The unemployment insurance benefits program is self-financing:
\[
\tau_u = \frac{\sum_{j=1}^{J} \sum_{a} \mu_j \lambda_j(a,s) = u \xi \omega \hat{h}}{\sum_{j=1}^{J} \sum_{a} \mu_j \lambda_j(a,s) = e \omega \epsilon \hat{h}} = \frac{\xi \sum_{j=1}^{J} \mu_j}{\sum_{j=1}^{J} \mu_j}.
\]  
(19)

viii. The lump-sum distribution of accidental bequests is determined by
\[
T^u = \sum_{j} \sum_{a} \sum_{s} \mu_j \lambda_j(a,s)(1 - \psi_{j+1}) A_j(a,s).
\]  
(20)

**MEASURE OF UTILITY**

In order to compare alternative social security arrangements, we need a measure of “average utility.” Given a policy arrangement \(\Omega = \{\theta, \xi, \tau, \tau_u\}\), we calculate
\[
W(\Omega) = \sum_{j=1}^{J} \sum_{a} \sum_{s} b^{-1} \left( \prod_{k=1}^{j} \psi_k \right) \lambda_j(a,s) U(C_j(a,s)),
\]  
(21)
as our measure of utility. \(W(\Omega)\) is the expected discounted utility a newly born individual derives from the lifetime consumption contingency plan \(C_j(a,s)\) under a given social security arrangement.
In order to obtain numerical solutions of the model and conduct welfare analysis, we need to choose particular values for the parameters of the model. We calibrate our model under the assumption that the model period is one year. The parameter values are shown in Table 1.

**Table 1**
**Parameter Values and Sources**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta_{i}$</td>
<td>Faber (1982)</td>
</tr>
<tr>
<td>$\mu_{j}$</td>
<td>$\frac{\psi_{j-1}}{1 + p}$ and $\sum_{j=1}^{J} \mu_{j} = 1.0$</td>
</tr>
<tr>
<td>$\epsilon_{j}$</td>
<td>Hansen (1991)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.012</td>
</tr>
<tr>
<td>$J$</td>
<td>65</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>45</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.45</td>
</tr>
<tr>
<td>$\xi$</td>
<td>0.40</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.64</td>
</tr>
<tr>
<td>$B$</td>
<td>1.3193</td>
</tr>
<tr>
<td>Initial output</td>
<td>1.0</td>
</tr>
<tr>
<td>Initial capital stock</td>
<td>3.0</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.08</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>2.00</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.011</td>
</tr>
</tbody>
</table>

**COMPUTING THE DECISION RULES**

The decision rules for each cohort are found by computing the following backward recursion starting from the last period of life at $j = J$ until the first age $j = 1$:

$$V_{j}(a_{j-1},s_{j}) = \max_{k,a} \{ U(c_{j}) + \beta \psi_{j+1} E_{j+1}(a_{j+1},s_{j+1}) \}$$  \hspace{1cm} (22)

subject to

$$a_{j} = (1 + r)a_{j-1} + q_{j} - c_{j} + T, \hspace{0.5cm} c_{j} \geq 0, \hspace{0.5cm} a_{j} \geq 0$$  \hspace{1cm} (23)

where

$$q_{j} = \begin{cases} (1 - \tau_{r} - \tau_{w}) \omega \xi \hat{h} & \text{for } j = 1, 2, \ldots, J^{*} - 1, \text{if } s = e; \\ \xi \omega \hat{h} / b & \text{for } j = 1, 2, \ldots, J^{*} - 1, \text{if } s = u; \\ (1 - \tau_{r} - \tau_{w}) \omega \xi \hat{h} / b & \text{for } j = J^{*}, J^{*} + 1, \ldots, J; \end{cases}$$  \hspace{1cm} (24)

and $\omega$ and $r$ are given by Eq. (12).

Our numerical solution method begins by discretizing the state space $D = \{d_{1}, d_{2}, \ldots, d_{m}\}$ where asset holdings are required to fall. The presence of borrowing constraints instructs us to choose the lower bound as zero: $d_{1} = 0$. The upper bound is found by experimentation, under the restriction that it have no impact on the individual’s decision rules. For the present problem, we set $d_{m}$ equal to 15, which is about 15 times the initial annual earning of an employed individual. The spacing was even between 0 and 15 with a total of 601 grid points. (We increased the number of grid points to 1,201 without changing the upper bound, but the numerical results were almost identical to the 601 case. Doubling the number of grid points raises the computational burden by a factor of 4.) Hence, the state space has 601 x 2 points for individuals below retirement age and 601 x 1 points for retired individuals. The control space is 601 x 1 for all individuals. It should be emphasized that, unlike the infinite-horizon case, no iterations are required to compute the decision rules. These rules
are computed from the single backward recursion shown above. In the infinite-horizon case, there is only one decision rule matrix that describes the behavior of all individuals, and that decision rule is obtained by iterating on the optimality equation. In the present finite-horizon case, sixty-five age-specific decision rule matrices, forty-four $601 \times 2$ matrices for individuals below the retirement age and twenty-one $601 \times 1$ matrices for retirees, must be computed and kept in core memory for subsequent use in calculating the distribution of agent types and aggregate variables. Also, a similar number of matrices with the same dimensions representing the distribution of agent types must be computed and kept in memory whereas the invariant distribution in the infinite-horizon case is a single matrix obtained by iterating on the rule given in the previous section. Therefore, the memory requirements of the finite-horizon case exceed those of the infinite-horizon case by an order of magnitude since a much larger number of matrices must be computed. However, as there are no iterations to perform in the finite-horizon case, but instead only two recursions, the overall CPU times are not very different from each other. A more specific comparison between the two cases is not very useful because not only are the underlying economic models quite different, but the dimensions of the state spaces are different as well.

Using the budget constraint Eq. (23) to substitute for $c_j$ in Bellman’s Eq. (22), the problem reduces to choosing the control variable $a_j$. We assume that $a_j \in D = \{d_1, d_2, \ldots, d_m\}$. For individuals at age $j^*$ or older, namely retirees, the state space is an $m \times 1$ vector $X = \{x = \omega \in D\}$. For individuals who are subject to idiosyncratic employment risk, at age $j^* - 1$ or younger, the state space is an $m \times 2$ matrix $X = \{x = (\omega, s) : \omega \in D, s \in S\}$. The control space for individuals of all ages is the $m \times 1$ vector $D$. For $j = j^*, j^* + 1, \ldots, J$, the decision rules take the form of an $m \times 1$ vector of asset holdings that solves the above problem. For $j = 1, 2, \ldots, j^* - 1$, the decision rules are $m \times 2$ matrices, one such matrix for each $j$, showing the utility maximizing asset holding for each level of beginning-of-period assets and employment status realization.

As death is certain beyond age $J(\psi_i = 0$ for $i > J)$, the value function at $J + 1$ is identically zero. Hence, the solution to

$$V_J(x_j) = \max_{\{c_j, a_j\}} \{U(c_j)\},$$

subject to

$$0 = (1 + r)a_{j-1} + b - c_j + T, \quad c_j \geq 0,$$

is an $m \times 1$ vector decision rule for age-$J$ individuals, $A_J$. Note that this is a vector of zeros because there is no bequest motive and death is certain after $J$. The value function at age $J$, $V_J$, is an $m \times 1$ vector whose entries correspond to the value of the utility function at $(1 + r)a_{j-1} + b + T$ with $a_{j-1}$ taking on the values $d_1, d_2, \ldots, d_m$. This value function $V_J$ is passed on to the next step, where the age-$(J - 1)$ decision rule and value function are calculated. The age-$(J - 1)$ decision rule is found by obtaining

$$V_{J-1}(x_{J-1}) = \max_{\{c_{J-1}, a_{J-1}\}} \{U(c_{J-1}) + \beta \psi_j V_J(x_j)\},$$

subject to

$$a_{J-1} = (1 + r)a_{J-2} + b - c_{J-1} + T, \quad c_{J-1} \geq 0, \quad a_{J-1} \geq 0.$$
syncratic employment risk. In fact, for \( j = 1, 2, \ldots, j^* - 1 \), disposable income can take one of two values: \((1 - \tau_s - \tau_u) \omega \epsilon_j h \) or \( \frac{1}{\epsilon_j} \omega h \), depending on the realization of \( s \). The decision rule for age \( j^* - 1 \) (and also for younger individuals) is an \( m \times 2 \) matrix describing the utility maximizing levels of asset holdings for each point in the state space \( X = D \times S \). Consequently, the value function \( V_{j^* - 1} \) is also an \( m \times 2 \) matrix.

For \( j = 1, 2, \ldots, j^* - 2 \), the optimality equation is given by

\[
V_j(\tilde{x}_j) = \max_{\{r,s\}} \left\{ U(c_j) + \beta \Phi_{j+1} \sum_s P(s',s)V_{j+1}(\tilde{x}_{j+1}) \right\},
\]

subject to

\[
a_j = (1 + r) a_{j-1} + q_j - \epsilon_j + T, \quad \epsilon_j \geq 0, \quad a_j \geq 0,
\]

where \( q_j \) is given by Eq. (24). For \( a_{j-1} = d_1 \) and \( s = e \), we search over \( a_j \in D \) that solves the above problem and report that value as the \( 1 \times 1 \) element of the \( m \times 2 \) decision rule \( A_j \). Then we search over \( a_j \in D \) for given \( a_{j-1} = d_1 \) and \( s = u \), and report the optimal value as the \( 1 \times 2 \) element of the decision rule for age \( j \). This process is repeated until all elements of the decision rule \( A_j \) are computed. This completes the computation of the decision rules \( A_j \) and value functions \( V_j \) for all ages; \( 2 \cdot (j^* - 1) \) matrices each \( m \times 2 \) and \( 2 \cdot (J - j^* + 1) \) vectors each \( m \times 1 \).

Given the large number of floating-point and integer calculations required to obtain the decision rules, we have used three devices in an attempt to reduce the computational burden without sacrificing much accuracy. We checked each of these devices for their contribution to computational errors relative to the original method and found that virtually no approximation error is introduced by these devices. First, we rely on the global (strict) concavity of the utility function and exit from the associated DO-loop in the code in which we record the first decline in expected utility. Second, we do not evaluate the utility function at each step in the solution procedure but, rather, follow Alvarez and Fitzgerald (1992) and use a linear interpolation procedure to compute the utility of a given permissible consumption level from the utility tabulation in the core memory (see footnote 1). Third, we break up the problem by making a preliminary pass over the coarse grid of asset values to determine an initial optimum. We then make subsequent searches over successively finer grids of asset values around the previous optimum. This saves us a large number of unnecessary searches. (This device also relies on the global strict concavity of the utility function.) The overall result is a reduction in CPU time from about 26.7 sec (to obtain the decision rules and the distribution of people) to just below 1.8 sec on the CRAY Y-MP/8-864 at San Diego. Alternatively, the same program takes about 25 sec to run on an IBM-compatible 486DX50 using Microsoft's 32-bit Fortran compiler. (Also, at the compile stage, we try to fully utilize vectorization either by writing the code accordingly or by selecting the aggressive vectorization options of Cray's or Microsoft's compilers.)

**OTHER UTILITY FUNCTIONS**

The parametric utility function in Eq. (6) has been criticized for its failure to accommodate some important features of the aggregate U.S. data on consumption and stock returns. As a result, some researchers have used different specifications for the utility function. These include time nonseparable utility functions that allow for habit persistence and durability in consumption, and recursive utility functions that violate some of the von Neumann–Morgenstern expected utility axioms. In this section, we will describe how our procedure can be used to solve models with different utility functions. However, it should be emphasized that much of this is conjectural, as we have not yet implemented it.

**Habit Persistence and Durability in Consumption:** Following Ferson and Constantinides (1991), let \( c_j^f \) denote the flow of consumption services at age \( j \) produced by current and past consumption expenditures (also see Dunn and Singleton, 1986; Eichenbaum, Hansen, and Singleton, 1988; Sundaresan, 1989; and Constantinides, 1990). Then the durability of consumption can be modeled by expressing the total flow of consumption at age \( j \) as \( c_j^f = \sum_{i=0}^{k} \alpha_i c_{j-i}, \) where \( \alpha_i \geq 0 \) and \( \sum_{i=0}^{k} \alpha_i = 1 \). The additional structure imposed by habit persistence can be described by the period utility function

\[
\left\{ c_j^f - h \sum_{n=1}^{k} \phi_n c_{j-n} \right\}^{1-\gamma} / (1 - \gamma),
\]
where $\phi_n \geq 0$, $\sum_{n=1}^{\infty} \phi_n = 1$, $h \geq 0$, and $\gamma$ is the coefficient of risk aversion or the reciprocal of the coefficient of intertemporal elasticity of substitution in consumption. The parameters $k$ and $k'$ denote the length of durability of the consumption good and the length of memory in habit persistence, respectively. This time nonseparable utility function is much too complicated to be of use in the present case unless the durability and memory parameters $k$ and $k'$ are small integers. One simplifying assumption is to restrict the utility function above so that it takes the general form $U(c_t, c_{t-1})$ where $U(\cdot, \cdot)$ is twice continuously differentiable, bounded, increasing in both $c_t$ and $c_{t-1}$, and concave in $(c_t, c_{t-1})$. In general, the optimal consumption plan will depend on the beginning-of-period asset holdings $a$ and previous consumption $c_t$. In particular, assume that for $j = 2, 3, \ldots, J$, the utility function is given by $U(c_j, c_{j-1}) = (c_j + \alpha c_{j-1})^{1-\gamma}/(1 - \gamma)$, and for $j = 1$ we have $U(c_1) = c_1^{1-\gamma}/(1 - \gamma)$. The parameter $\alpha$ is a measure of durability if it is positive, and a measure of habit persistence if it is negative. In this case, the state space expands to include the permissible values of consumption in the previous age. In principle, our method can be used to solve this version, but the curse of dimensionality poses a serious problem.

"The parameter $\alpha$ is a measure of durability if it is positive, and a measure of habit persistence if it is negative. In this case, the state space expands to include the permissible values of consumption in the previous age. In principle, our method can be used to solve this version, but the curse of dimensionality poses a serious problem."
of previous age consumption. The backward recursion follows

$$V_j(\tilde{x}_j) = \max_{\{c_j,a_j\}} \left\{ U(c_j, a_{j-1}) + \beta \psi_{j+1} \sum_i \Pi(s', s)V_{j+1}(\tilde{x}_{j+1}) \right\},$$

subject to the period-$j$ budget constraint. Note that $\tilde{x}$ is now equal to $(a, c_{-1}, s)$. 

**Recursive Utility:** In almost all applied and theoretical work, researchers have used utility functions that maintain the complete set of von Neumann–Morgenstern axioms. Kreps and Porteus (1978, 1979), on the other hand, drop one of the axioms and allow the timing of resolution of uncertainty to matter. (See Weil [1990], Kocherlakota [1987], and Epstein and Zin [1987], among others, for application of recursive preferences to some popular puzzles in macroeconomics.) A desirable feature of this recursive utility specification is the breakup of the inverse relationship between the coefficient of risk aversion and the coefficient of intertemporal elasticity of substitution. In the context of the overlapping generations model used here, we can describe the use of recursive utility as follows. Suppose that the utility function at age $j$ is given by $W_j = U(c_j)$, say $W_j = c_{-j}^{1-\gamma}(1 - \gamma)$. For $j = 1, 2, \ldots, J - 1$, the utility function is assumed to be $W_j = U(c_j, E_j W_{j+1})$, say $W_j = \beta W_{j+1}^{1-\gamma}(1 - \gamma)$. The generalized isoelastic preferences above follow from Weil (1990). Here, $\beta$ is the subjective discount factor only under certainty. Under uncertainty, the subjective discount factor is endogenous and age-specific and is given by $\beta W/\partial c$ evaluated at the optimal consumption plan. The parameter $\gamma$ is the constant coefficient of risk aversion for static gambles. (Under certainty this parameter drops out of preference specification, as Weil [1990] points out, because risk aversion is irrelevant for ranking deterministic consumption sequences.) The (constant) intertemporal elasticity of substitution for deterministic consumption paths is given by $1/\rho$. In this setup, an individual prefers early resolution of uncertainty if $\gamma$ exceeds $\rho$. Our method can be applied in this case as follows. The period-$j$ problem is the same as before:

$$V_j(x_j) = \max_{\{c_j,a_j\}} \left\{ U(c_j) \right\},$$

subject to the period-$j$ budget constraint; $x = a$. To evaluate $W_{j-1} = U(c_{j-1}, E_{j-1} W_j)$, we use $E_{j-1} W_j = \beta \psi_{j} \sum_i \Pi(s', s) W_j$, but since there is no idiosyncratic income risk during retirement, this is simply $\beta \psi_{j} W_j$. Now the period-$(J - 1)$ problem is written as

$$V_{J-1}(x_{J-1}) = \max_{\{c_{J-1}, a_{J-1}\}} \left\{ W_{J-1} + \beta \psi_{J} V_{J}(x_{J}) \right\},$$

subject to the period-$J$ budget constraint. Note that the first term on the right-hand side of the above problem is evaluated using the complicated parametric utility function specification above and the rule $E_{j} W_{j+1} = \beta \psi_{j+1} \sum_i \Pi(s', s) W_{j+1}$. We then proceed as before and compute the backward recursion using

$$V_j(\tilde{x}_j) = \max_{\{c_j,a_j\}} \left\{ U(c_j, \beta \psi_{j+1} \sum_i \Pi(s', s) W_{j+1}) + \beta \psi_{j+1} \sum_i \Pi(s', s) V_{j+1}(\tilde{x}_{j+1}) \right\},$$

subject to the period-$j$ budget constraint. Note that the use of recursive utility does not add much to the memory requirements, but obviously it can add to the floating-point calculations through the additional and more complicated value function calculations. However, tabulating this function once in the beginning and then using linear interpolation at each step in the search for an equilibrium should add only trivially to the overall computational burden.

**Computing the Age-Dependent Distribution of Agent Types**

To obtain the distribution of agents, $\lambda_j(a,s)$, into beginning-of-period asset holding levels and employment categories, we start from a given initial wealth distribution $\lambda_1$. The choice of $\lambda_1$ will influence the equilibrium of the model. We assume that newborns have zero asset holdings, so $\lambda_1$ is taken to be an $m \times 2$ matrix with zeros everywhere except the first row, which is equal to (0.94, 0.06), the expected employment and unemployment rates, respectively. The distribution of agents at the end
of age 1, or equivalently, at the beginning of age 2, is found by

\[ \lambda_2(a',s') = \sum_{s} \sum_{a:a' \in A_2(a,s)} \Pi(s,s')\lambda_1(a,s). \]

We implicitly assume that all members of a given cohort face the same mortality risk regardless of their wealth. Hence, the distribution does not change between ages when premature death occurs.

Starting from the initial wealth distribution \( \lambda_1 \), some individuals will be employed and some will be unemployed at age 1. Depending on the realization of the employment status, individuals will choose asset holdings using the decision rules described above. Therefore, at the beginning of age 2, they will go to (possibly) different points in the state-space matrix \((a,s)\). Each entry in the \( m \times 2 \) matrix \( \lambda_2 \) gives the fraction of 2-year-old agents at that particular combination of asset holdings (chosen at the end of the age-1 optimization problem) and period-2 employment status. Note that, for each \( j \), each element of \( \lambda_j \) is nonnegative, and the sum of all entries equals 1.

In general, given \( J \) decision rules \( A_j \) and an initial income distribution \( \lambda_1 \), the age-dependent distributions are computed from the forward recursion

\[ \lambda_j(a',s') = \sum_{s} \sum_{a:a' \in A_j(a,s)} \Pi(s,s')\lambda_{j-1}(a,s). \]

Note that for \( j = j^*, j^* + 1, \ldots, J \), \( \lambda_j \) is \( m \times 1 \) because the retired individuals are not subject to idiosyncratic employment risk.

**THE NUMERICAL ALGORITHM**

Having computed the decision rules and the measures of agent types for all cohorts, one needs to check to see if the candidate solution constitutes an equilibrium. Note that the equilibrium described here amounts to finding the fixed point of an operator implicitly defined over the aggregate asset holdings \( K \), as the lump-sum bequest distribution \( T \) is a monotone function of the capital stock. Hence, the numerical algorithm focuses on obtaining convergence in the aggregate capital stock.

Step 1. Choose the convergence criterion \( \epsilon > 0 \). We chose \( \epsilon = 0.001 \). This criterion is obtained through
experimentation. A smaller $\epsilon$ increases the number of iterations whereas a larger $\epsilon$ may change the results significantly.

**Step 2.** Start with a candidate steady-state equilibrium aggregate capital stock $K^{(0)}$ and the lump-sum unintended bequest distribution $T^{(0)}$. Compute aggregate labor input $N = 0.94k \sum_{t=1}^{\tau} \mu_t$. Use the first-order conditions from the firm's profit maximization problem to obtain the corresponding guesses for the relative factor prices $w$ and $r$, and substitute these in the individual's budget constraint.

**Step 3.** Compute the decision rules for each cohort by completing a backward recursion.

**Step 4.** Compute the distribution of agent types for each cohort by completing a forward recursion.

**Step 5.** Using Eq. (15), compute the new aggregate capital stock $K^{(1)}$.

**Step 6.** Check to see if $K^{(1)}$ is sufficiently close to the initial guess $K^{(0)}$; that is, check whether $|K^{(1)} - K^{(0)}| < \epsilon$. If not, compute $K^{(2)} = (K^{(0)} + K^{(1)})/2$ and $T^{(2)} = (T^{(0)} + T^{(1)})/2$, set $K^{(0)} = K^{(2)}$ and $T^{(0)} = T^{(2)}$, and go to step 2.

**Step 7.** Compute aggregate consumption, investment, and output using the decision rules, distribution of agent types, and the population shares of cohorts, and check whether the commodity market clearing condition given by Eq. (16) is approximately satisfied. Note that this is merely a check on the internal consistency of the model and the accuracy of the code that performs the computations. When the decision rules and the distribution of the agent types are computed correctly, and the aggregate variables are calculated correctly, the market clearing condition ought to hold in equilibrium as it is a weighted average of the individuals' budget constraints. If the problem is correctly specified and the code is accurate, excess demand is typically less than 0.0001 when the capital stock converges. If excess demand is sufficiently small when the aggregate capital stock converges, then go to step 8. If not, check the code for accuracy or the economic model for internal consistency.

**Step 8.** Stop.

### Numerical Findings

This section presents some numerical results from the application of the numerical algorithm described in the previous section. Each row in Table 2 represents a different steady-state equilibrium. The first column gives the social security benefit level, and the remaining columns contain the equilibrium values of aggregate variables and the average utility associated with the corresponding arrangement. An increase in the benefit level monotonically reduces the capital stock and consequently raises the net real return to capital. Given that the output elasticity of capital is less than one, the capital-output ratio also falls as the benefit level rises.

In the experiment reported in Table 2, the discount factor is set equal to 1.011, corresponding to a negative rate of time preference (see Imamoglu, Imamoglu, and Joines [1992] for a discussion of the choice of the parameters, including the discount factor, and a detailed sensitivity analysis using a coarse grid of parameter values around the ones that are used in the benchmark economy). In this experiment, the optimal social security replacement rate is 30%. The strong saving motive represented by a negative rate of time preference leads to a large capital stock. In the absence of social security, this economy is dynamically inefficient in the sense of Diamond (1965), and social security provides a higher rate of return than physical capital. In addition to providing a higher return than physical capital, social security reduces private saving, and a replacement rate of between 10% and 20% eliminates the over-

---

3. An alternative to this ad hoc convergence procedure is to use Newton's method or a safeguarded procedure for finding the zero of the excess demand function. However, we believe there are a number of good reasons for favoring discretion over rule in this case. Among them are (1) the possibility of making an error or two in specifying the initial parameters of preferences and technology and some institutional features like population growth, mortality risk, etc., which may create nonzero excess demand even when the aggregate capital stock converges, (2) unnecessary iterations over the aggregate capital stock when the aggregate utility measure is fairly flat, and (3) unknown functional dependence of some aggregate variables like accidental bequests on the aggregate capital stock since accidental bequests depend on the equilibrium distribution of people as well.
Table 2
Population Growth and Lifetime Uncertainty, $\beta = 1.011, \gamma = 2$

<table>
<thead>
<tr>
<th>Tax Rate</th>
<th>Wage Income</th>
<th>Return to Capital</th>
<th>Average Consumption</th>
<th>Capital Stock</th>
<th>Average Income</th>
<th>Average Utility</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.0000</td>
<td>1.0064</td>
<td>0.0041</td>
<td>0.7396</td>
<td>5.2241</td>
<td>-97.859</td>
</tr>
<tr>
<td>0.10</td>
<td>0.0203</td>
<td>0.9725</td>
<td>0.0093</td>
<td>0.7421</td>
<td>4.7505</td>
<td>-96.293</td>
</tr>
<tr>
<td>0.20</td>
<td>0.0406</td>
<td>0.9434</td>
<td>0.0143</td>
<td>0.7422</td>
<td>4.3651</td>
<td>-95.476</td>
</tr>
<tr>
<td>0.30</td>
<td>0.0609</td>
<td>0.9170</td>
<td>0.0192</td>
<td>0.7409</td>
<td>4.0599</td>
<td>-95.175</td>
</tr>
<tr>
<td>0.40</td>
<td>0.0812</td>
<td>0.8951</td>
<td>0.0235</td>
<td>0.7382</td>
<td>3.7723</td>
<td>-95.339</td>
</tr>
<tr>
<td>0.50</td>
<td>0.1015</td>
<td>0.8760</td>
<td>0.0276</td>
<td>0.7352</td>
<td>3.5750</td>
<td>-95.801</td>
</tr>
<tr>
<td>0.60</td>
<td>0.1218</td>
<td>0.8583</td>
<td>0.0315</td>
<td>0.7318</td>
<td>3.3576</td>
<td>-96.548</td>
</tr>
</tbody>
</table>
| 1.00     | 0.2030      | 0.8014            | 0.0460              | 0.7163        | 2.7731        | -101.570       

However, a newly born individual would prefer a social security arrangement with the higher replacement rate of 30% over alternative arrangements. This higher optimal replacement rate arises because social security imperfectly substitutes for (missing) private annuity contracts.

Figure 1 shows age-income and age-consumption profiles from the benchmark model economy using a replacement rate of 30%. Average income rises with age and starts to fall at about mid-working age, reflecting the earnings profile. After retirement, income is constant. The path of average consumption is smoother than the income path and starts to decline at about the retirement age or shortly thereafter.

Figure 2 shows asset profiles for selected social security replacement rates in our benchmark economy. In all three cases, the asset profile rises during working ages and starts to fall after retirement. Assets are completely exhausted at death. Asset holdings are negatively related to the replacement rate because, at low replacement rates, individuals are motivated to save not only to insure against the idiosyncratic employment risk but also to provide for old-age consumption.

Figures 3 and 4 characterize the distribution of wealth in our benchmark model economy. Because all agents are born with identical endowments of both human and physical capital, our model cannot generate the degree of heterogeneity in wealth observed in actual data. Within-cohort heterogeneity is due entirely to dif-
ferences in employment histories across agents. However, most of the heterogeneity in asset holdings occurs across rather than within cohorts. Figure 3 shows the overall distribution of assets. Note that a large portion of individuals, about 18%, hold very little wealth. This is a consequence of our assumption that all newborns start life with zero assets, so that most of the "poor" individuals in our model economies are young. Figure 4 shows the distribution of wealth for selected age groups. Age-44 individuals, who are in the final year of possible employment, are the wealthiest among the four age groups considered. As individuals grow older, they run down their private assets, and at age 60, five years before the maximum age, the asset distribution is very narrowly peaked at a low asset level.

**Concluding Remarks**

This article describes an incomplete markets model and outlines a numerical solution algorithm to compute its steady-state equilibrium. The model departs from the Arrow-Debreu world of complete contingent claims markets by exogenously assuming the presence of borrowing constraints. An infinite-horizon setup is described, as it provides the basic heterogeneous-agent framework. A detailed description of an overlapping generations version of the incomplete markets model is provided. Solution methods that rely on approximations around some reference point or those that use the Euler equations in an interior solution are not suitable for these models. Rather, a numerical solution algorithm, which has been successfully applied in a variety of settings, is outlined here. The algorithm is implemented using the Fortran compilers on the Minnesota CRAY Y-MP and San Diego CRAY X-MP.

The numerical solution algorithm outlined here has been used by İmrohoroğlu, İmrohoroğlu, and Joines (1992) in a study of social security. Their analysis shows that in an incomplete markets model, the welfare analysis of social security is quite different from that in an Arrow-Debreu environment. The imperfect insurance setup and the solution algorithm outlined here are likely to be useful in addressing other life-cycle issues as well.
ACKNOWLEDGMENT

The authors are grateful to the guest editor and two anonymous referees for their comments.

This material is based upon work supported by the National Science Foundation under Grant No. SES-9210291.

BIOGRAPHIES

Ayse Imrohoroglu is an assistant professor of finance and business economics at the University of Southern California. She received a B.S. degree from the Middle East Technical University in Turkey and a Ph.D. degree from the University of Minnesota. Her research interests are in the fields of macroeconomics, monetary economics, and labor economics. Her papers have appeared in the Journal of Economic Dynamics and Control, Journal of Money, Credit, and Banking, and Journal of Political Economy.

Selahattin Imrohoroglu is an assistant professor of finance and business economics at the University of Southern California. He received a B.S. degree from the Middle East Technical University in Turkey and a Ph.D. degree from the University of Minnesota. His research interests are in the fields of macroeconomics, public finance, monetary economics, and econometrics. His papers have appeared in the Journal of Economic Dynamics and Control and Journal of Money, Credit, and Banking.

Douglas H. J. Joines is an associate professor of finance and business economics at the University of Southern California. He received an A.B. degree from Duke University and M.B.A. and Ph.D. degrees from the University of Chicago. His research interests are in the fields of macroeconomics, monetary economics, and public finance. His papers have appeared in the American Economic Review, the Journal of Business, the Review of Economics and Statistics, Economic Inquiry, the Journal of Monetary Economics, the Journal of International Money and Finance, the Journal of Business and Economic Statistics, and other publications.

SUBJECT AREA EDITOR

John Rust

REFERENCES


Imrohoroglu, A., Imrohoroglu, S., and


