The Broadest Necessity

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Abstract

In this paper we explore the logic of broad necessity. Definitions of what it means for one modality to be broader than another are formulated, and we prove, in the context of higher-order logic, that there is a broadest necessity, settling one of the central questions of this investigation. We show, moreover, that it is possible to give a reductive analysis of this necessity in extensional language (using truth functional connectives and quantifiers). This relates more generally to a conjecture that it is not possible to define intensional connectives from extensional notions. We formulate this conjecture precisely in higher-order logic, and examine concrete cases in which it fails. We end by investigating the logic of broad necessity. It is shown that consistently with higher-order logic, the logic of broad necessity can be anywhere between $\mathsf{S4}$ and Ver; we give some reasons to think that it is strictly weaker than $\mathsf{S5}$.

Say that a necessity operator, $\Box_1$, is *at broad as* than another, $\Box_2$, if every instance of the following schema[1] holds:[2]

(*) $\Box_3(\Box_1 A \to \Box_2 A)$

Here $\Box_3$ can be substituted for any other necessity operator and A for any sentence. For one operator to be broader than another, then, it must strictly imply the other relative to any candidate notion of strict implication. Note, by contrast, that some authors (e.g. Hale [19]) say that an operator is broader than another if it *entails* the other, where entailment is strict implication according to the broadest kind of necessity. However to appeal to such a notion would be to prejudge an important question — whether there *is* a broadest necessity — and moreover would make it hard to spell out what it means to be 'the broadest necessity' in a non-circular way.

In order for this definition to be useful we need some characterization of the sorts of operators that can be substituted for $\Box_3$. In other words, we want to know: What is a necessity operator? A natural thought is that in order for an operator to count as a candidate necessity operator it must be *normal* in the sense of Kripke [25]. However, this is again a hard idea to spell out without presupposing the broad notion of entailment.[3] In what follows, then, I will employ a couple of weaker notions. The first is that of an operator that applies to a tautology:

[1] Here and elsewhere I understand 'schema' in such a way that universal closures of instances count as instances.

[2] The use of the terminology of 'broadness' has its pitfalls: the broader an operator, the fewer propositions it applies to. The terminology derives from a way of modeling necessity operators in terms of worlds: the broader the operator the broader the set of worlds it quantifies over.

[3] An operator $\Box$ (as opposed to an operator expression) might be said to be normal only if it $\Box A$ is *logically necessary* (propositionally entailed by the empty set of propositions) whenever A is, and that the proposition $\Box(A \to B) \to \Box A \to \Box B$ is logically necessary.

> **Weak Necessity**: $\Box$ is a *weak necessity* operator if and only if $\Box\top$.

Clearly metaphysical necessity passes the test, as do various sorts of causal and epistemic necessity. Indeed, every normal modal operator passes the test: the 'always' operator from tense logic, the determinacy operator employed in the study of vagueness and counterfactual necessity (a proposition is counterfactually necessary when its negation counterfactually implies a contradiction) all pass my test for being a weak necessity operator.[4] Some operators pass the test that are not as theoretically interesting: for example if Jess said a tautology then 'Jess said that' counts as a weak necessity operator in my sense as well. Due to this weakness it will sometimes useful to consider a stronger notion as well:

> **Necessity**: $\Box$ is a *necessity* operator if and only if $\Box'\Box\top$ whenever $\Box'$ is a weak necessity operator.[5]

A weak necessity operator applying to $\top$ might do so only contingently (for instance, the operator 'Jess said that'). By contrast a necessity operator must apply to $\top$ necessarily, for any reasonable candidate notion of necessity. Since any reasonable candidate necessity operator will at least pass the test for being a weak necessity our definition ensures this. Again, the candidates listed above are arguably strong necessities in this sense, given certain assumptions about fineness of grain.[6] Note, in particular, that one important property of all normal modal operators can be proven from this definition. If $\Box$ is a necessity, it is also a weak necessity operator: since 'it's true that' is a weak necessity operator, truth must apply to $\Box\top$ (by Strong Necessity). Since $\Box$ is a weak necessity operator it also follows from the definition that $\Box\Box\top$ must be true, and by repeating this reasoning that $\Box\Box\Box\top$, and so on. Indeed, it can be show that the iterations of $\Box$ — $\Box\Box$, $\Box\Box\Box$ and so on — are themselves also necessity operators by a similar sort of argument.

It is fair to wonder whether the definition of broadness and necessity operator adopted above can do the same work as the standard definitions, couched in terms of propositional entailment or some affiliated notion. Our definition ensures that if $\Box_1$ is broader than $\Box_2$, then the former 'entails' the latter, on each proposition, for every candidate notion of 'entailment', where these candidates are captured by our definition of a necessity operator. One might worry that this definition of a candidate casts its net too wide, and nothing can entail anything else in all the relevant senses. We shall see shortly that given certain assumptions about fineness of grain this worry is ill founded. What matters for our purposes is that we have caught all the kinds of entailment that matter, for if $\Box_1$ is broader than $\Box_2$ then $\Box_1 A$ metaphysically implies $\Box_2 A$, always implies it, determinately implies it, *a priori* implies it, and so on.

In the above $\top$ stands for a tautology. However, for all we've said, our definition could depend on which tautology we use. If propositions are very fine grained, for example, then there could be operators that apply to some tautologies but not others. In this paper

[4] Normal modal operators satisfy the smallest modal logic K, which has a rule of necessitation: if you can prove A from K then you can prove $\Box A$. Any normal modal operator applies to $\top$ because it is provable in K.

[5] That is, '$\Box$' expresses a necessity just in case every instance of a certain schema is true: $\Box'\top \to \Box'\Box\top$. To make this idea precise, without abusing the use-mention distinction, we would need to quantify into operator position: something we will introduce in section 3.

[6] The relevant assumption is that provable equivalence in certain systems suffices for identity (a version of the Rule of Equivalence discussed in section 2). For given the standard logic of metaphysical necessity, the proposition that $\top$ is metaphysically necessary is provably equivalent, and thus identical to $\top$. So any weak necessity (any operator applying to $\top$) must also apply to the proposition that $\top$ is metaphysically necessary by Leibniz's law. The same reasoning applies for 'always', 'determinately' and so on.

we will rule this possibility out by adopting the assumption of *Booleanism*. We will have more to say about this later, but for the present purposes this assumption can effectively be summarized by the idea that we can substitute sentences for other sentences that are provably equivalent in the propositional calculus.[7] In particular, it follows that $\Box\top$ is true only if $\Box\top'$ for any two tautologies $\top$ and $\top'$, because all tautologies are provably equivalent in the propositional calculus. (Note: Booleanism is not to be confused with the thesis that metaphysically necessarily equivalent propositions are the same, which is a much stronger thesis that will be rejected here.)

An open question at this juncture is whether there is such a thing as a maximally broad necessity operator: a necessity operator such that no other necessity operator is broader than it. One way this could fail would be if for every necessity operator there was a necessity operator broader than it. Even if this is settled, there could be two or more maximally broad necessity operators such that neither is broader than the other (indeed, Dorothy Edgington has defended a view of exactly this sort [13]). Both of these possibilities preclude the existence of a *broadest* necessity operator: a necessity operator that is at least a broad as every other necessity operator.

In the wake of Kripke's 'Naming and Necessity' this question might seem like a particularly urgent one. Once it is acknowledged that metaphysical necessity and *a priori* truth are different notions with different extensions there is a natural question as to which of these is broader than the others, if any. Once this question is raised it is tempting to ask similar questions about other cognate notions such as analyticity, logical validity, and so on. But it is important to distinguish here two different sorts of things one might be tempted to call a 'kind of necessity'. Notions like analyticity, logical validity (truth in all models) and provability are all properties of sentences, whereas notions like metaphysical necessity and (more contentiously) *a priori* truth are properties of propositions. Since no linguistic notion of necessity applies to a propositional necessity, and conversely (assuming that no propositions are sentences and conversely) the question of broadness across these sorts of necessity is not particularly interesting. Indeed, the question of whether there is a broadest linguistic necessity is also trivialized in the present setting: sentences are so fine grained that the assumption of Booleanism doesn't apply. The linguistic analogues of our above definitions make little sense without this assumption.[8] In short, the question for linguistic necessities would have to be radically reformulated, and I suspect it will be hard to do so in a suitably precise setting to derive the sorts of results I present below.

This existence of a broadest necessity operator is the topic of this paper. I shall argue, along with many others (e.g. Hale [19], McFetridge [28]), that there is a broadest necessity. But my project will also be to give an *analysis* of this necessity; indeed an analysis that has a reasonable claim to being a reductive one.

The paper is structured as follows. In section 1 I draw out some general features of the

[7]Technically, by Booleanism I shall mean the rule that if A and B are provably equivalent given the propositional calculus *and* Booleanism, then A and B are intersubstitutable (this is not viciously circular, it has the same status as the rule of necessitation in modal logic). This, for example, allows us to substitute $\Box(A\wedge B) \to \Box(B\wedge A)$ for $\top$ — something which couldn't be proved from the intersubstitutivity of tautological equivalents alone.

[8]For example, the linguistic analogue of our definition of a necessity operator would be dependent on which tautology was chosen in the definition. The question of broadness on this way of doing things would become uninteresting. Suppose that we chose the tautology $A \vee \neg A$. Then following parallel definitions, a linguistic necessity predicate N_1 is broader than another N_2 iff $N_3(\ulcorner N_1(\ulcorner B\urcorner) \to N_2(\ulcorner B\urcorner)\urcorner)$ is true for every every sentence B and predicate N_3 applying to $\ulcorner A \vee \neg A\urcorner$. But any predicate that applies to $\ulcorner A \vee \neg A\urcorner$ and nothing else can be substituting for N_3, but does not apply to $\ulcorner N_1(\ulcorner B\urcorner) \to N_2(\ulcorner B\urcorner)\urcorner$. So no linguistic predicate would count as broader than any other if we followed analogues of the above definitions.

logic of the broadest necessity on the supposition that it exists, and give some reasons to think that metaphysical necessity is not the broadest necessity. In section 2 we investigate a definition of necessity explored by Suszko [44] and Cresswell [9], in which it is explained in terms of another intensional connective — propositional identity. The main results of the paper are found in section 3, where we introduce the framework of higher-order logic, and give a definition of the broadest necessity in logical vocabulary, using only the truth functional connective $\rightarrow$ and the higher-order quantifiers $\forall$. From this we will be able to *prove* from this definition that this operator is the broadest necessity, and are also able to prove that it is identical to the Cresswell-Suszko operator. It's worth emphasizing that this result might seem initially somewhat surprising, since by contrast it is clearly not possible to define any intensional notions from the truth functional connectives and the first-order quantifiers. We explore this fact, introducing explicit definitions of extensional and intensional contexts, and discuss the result that it's possible to define intensional contexts from extensional ones. We also draw some connections between substantive principles in higher-order logic and the logic of broad necessity; we see for example, that the axiom of functionality implies the Barcan formula, and prove uniqueness theorems for identity and broad necessity. Finally we turn to the question of the logic of the broadest necessity in section 4, and give some tentative reasons to think that it might be weaker than $\mathsf{S5}$ (but no weaker than $\mathsf{S4}$). In the appendix we establish that $\mathsf{S4}$ is complete relative to a general class of models of higher-order logic, establishing that the axiomatic system of higher-order logic of section 3 cannot prove anything about the propositional modal logic of the broadest necessity that cannot already be proven in $\mathsf{S4}$. $\mathsf{S5}$, by contrast, is sound and complete with respect to full models.

1 Metaphysical Necessity

Let us for a moment examine what follows from the supposition that there *is* a broadest necessity operator, L. It is natural to wonder what modal principles this operator would obey. Consider first the operator that maps each proposition to itself. (Later we will see that this operator can be defined in higher-orderese by the formula $\lambda X\, X$ where X is a variable taking the position of a sentence; for now we may assume it as a primitive.[9]) We shall call this the truth operator and we shall abbreviate it as 'T'. Since $T\top$ is just the same proposition as $\top$, $\Box T\top$ is true whenever $\Box\top$ is true. This means that T is a necessity operator. If L is the broadest necessity it is broader than T. So substituting T for both $\Box_3$ and $\Box_2$ in (*) we get $T(LA \rightarrow TA)$ which is just equivalent to the T axiom:

T $LA \rightarrow A$

This means that the broadest necessity, if there is one, must be factive. Intuitively this is just because the broadest necessity operator must at least be broader than truth.

Given any proposition B, we may define a complex operator $K := \lambda X\,(L(X \rightarrow B) \rightarrow LB)$. That is, the operator that applies to A iff: if A L-necessitates B then B is L-necessary. This operator is, in fact, a necessity operator: $K\top$ is just $L(\top \rightarrow B) \rightarrow LB$. By Booleanism we may substitute $\top \rightarrow B$ for B, to obtain the equivalence of $K\top$ with the tautology $LB \rightarrow LB$, since $\top \rightarrow B$ and B are equivalent in the propositional calculus. Given that $K\top$ is equivalent to a tautology (namely, $LB \rightarrow LB$), every weak necessity applies to $K\top$

[9] Although given Booleanism it is also definable as double negation.

(again applying Booleanism) so K is a necessity operator.[10] If L is the broadest necessity operator it must be as broad as this operator, so by substituting K for $\Box_2$, and T for $\Box_3$ in (*) as before, we get: $LA \to (L(A \to B) \to LB)$, which rearranges to an instance of the familiar modal principle K:

K $L(A \to B) \to (LA \to LB)$

Since we can construct a similar operator for any choice of B, and argue in a parallel way that L must be broader than it, the K schema holds in full generality.

The preceding results hold even on the weaker assumption that L is a broadest weak necessity. However, given that L is the broadest strong necessity then it follows from our earlier observation that its iterations LL, LLL, and so forth, are also necessities. This entails two things. Firstly, if L is the broadest necessity it must in particular be at least as broad as LL:

4 $LA \to LLA$

this follows by substituting T for $\Box_3$ and LL for $\Box_2$ in (*). Note finally that T, K and 4 are all instances of (*) with $\Box_3$ replaced with the truth operator. If instead we replace $\Box_3$ with L, we get the L-necessity of each of T, K and 4, and indeed given the 4 principle we can infer that these principles are L-necessary at every order — for example, from $L(LA \to LLA)$ we can infer $LL(LA \to LLA)$, $LLL(LA \to LLA)$, and so on ad infinitum. The resulting system is easily seen to be equivalent to the modal system S4, since the result of necessitating all the axioms at all orders has the same effect as assuming the law of necessitation.

Thus we have argued that if there is a broadest necessity operator its logic is at least S4.[11] But the question of whether there is a broadest necessity operator is left open. The orthodoxy, which I wish to uphold, is that there is a broadest such necessity. However the predominant strategy towards answering this question in the affirmative has been to argue that a particular candidate fills that role: *metaphysical necessity* (see, e.g., Kripke [26]).[12] I think there are several reasons to be dissatisfied with this candidate.

As Kripke demonstrated, there are some metaphysically necessary truths that are not *a priori* — for example the truth that it's actually sunny is necessary (if true) but certainly not *a priori*. So metaphysical necessity is not broader than knowability *a priori*. Moreover, *a priori* necessity is not the broadest necessity operator either, because there are *a priori* that aren't metaphysically necessary, such as the truth that it's actually sunny if and only if it's sunny. It appears as though we have two candidate operators, but they are incomparable — neither is broader than the other.[13]

This case is contentious because one might think that *a priori* knowability, being explained in terms of intentional attitudes like knowledge, is subject to a sort of guise sensitivity that infects many attitude verbs, and that the proposition that it's actually sunny is knowable *a priori* if it is accessed via the right sort of guise.[14] One might argue that *a priori*

[10]This argument is tricky in that it uses the sort of iterated application of Booleanism noted in footnote 7.

[11]And by an argument due to Scroggs this logic cannot be stronger than S5: see the discussion in section 4.

[12]In Kripke's words, metaphysical necessity is 'necessity in the highest degree' (p99).

[13]Edgington [13], for example, concludes from this that there are just two independent families of modal notions — metaphysical modalities and epistemic modalities; see McFetridge [28] for critical discussion.

[14]On a simple model there is only one metaphysically necessary truth, which both 'it's actually sunny' and '1=1' both express, and that this proposition is knowable *a priori* if accessed via a guise corresponding to the latter sentence.

knowability is thus not a property of propositions alone, and so is closer to the linguistic notions of necessity we set aside earlier.

The question of whether the puzzles surrounding propositional attitudes force there to be a necessity operator that is broader than metaphysical necessity is fraught. If there is a necessary proposition, p, such that one could believe that $\top$ (say) without believing that p then metaphysical necessity would not be broader than belief (the latter is a weak necessity operator because by supposition it applies to $\top$). Yet there are many maneuvers available for explaining why it might *seem* as though you could believe a tautology without believing every necessary proposition, even if you *can't*.[15]

Given certain background assumptions, however, there are independent sources of fineness of grain that have nothing to do with propositional attitudes, from which broader notions of necessity can be introduced. One of these sources of fineness of grain comes from the presence of tense operators, given the assumption of temporalism (that there are some propositions that are not always true or always false), and some reasonable theses about the logic of 'always'.[16] On the orthodox treatment of tense operators (see Kaplan [23] and Fine [16]) it's a temporary matter which propositions are necessary. For example, suppose it is Wednesday. Since it is Wednesday it is actually Wednesday, and moreover, since whatever is actually true is necessarily actually true, it is necessarily actually Wednesday. But of course, it is not always Wednesday, and so it is not always actually Wednesday. Thus we have a proposition A — that it's actually Wednesday — that is necessary but not always true. In short the following instance of (*) has false instances, and so metaphysical necessity cannot be broader then the always operator.

If it's metaphysically necessary that A then it's always the case that A.

If the invocation of the actuality operator seems too abstruse, the point can be made without appealing to them. It is natural to think that while non-fundamental facts, like the fact that it's Wednesday, can be temporarily true, all *fundamental* facts are eternal. Facts about the field values at particular space-time points, for example, are always true if ever true. Moreover it's natural to think that all truths supervene, metaphysically, on such fundamental truths. Let F be the conjunction of all the fundamental truths. Since F is true, and it's Wednesday we know that it's metaphysically possible that both F and it's Wednesday. Moreover, since it won't be Wednesday tomorrow, but it's always the case that F (since F is fundamental), it follows that sometimes F and it's not Wednesday. But if metaphysical necessity were broader than eternal truth, then 'sometimes F and it's not Wednesday' entails 'possibly F and it's not Wednesday'. This contradicts a natural thesis stating that everything metaphysically supervenes on the fundamental, for we have shown there are two metaphysical possibilities agreeing about the fundamental facts (assuming that the same propositions are fundamental on Wednesday as on any other day) but disagreeing about whether it is Wednesday.[17]

[15] There are many such strategies that could be used to do this. See, for example, Stalnaker [42], Salmon [37], Soames [41], Saul [38] (p6), Crimmins & Perry [10], Richard [35], Braun [6] and so on.

[16] One model of temporalism identifies propositions with sets of world-time pairs. On this model, propositions are more fine grained than sets of worlds, and so one would not expect propositions to be individuated by necessary equivalence. That is, one would expect to be able to find metaphysically necessarily equivalent propositions that are distinct. In particular if there was a metaphysically necessary proposition, p, that was distinct from $\top$ we could in principle find an operator O applying to $\top$ but not p. By (*) this would mean that metaphysical necessity was not broader than O. This is the rough intuition at any rate; we iron out the details in what follows.

[17] The fact that metaphysical necessity is not broader than eternal truth is of course a surprising conse-

Vagueness is another context in which we employ operators that are not subsumed by metaphysical necessity. If Harry is bald, and it is moreover not borderline whether Harry is bald we say that it is *determinate* that Harry is bald. In the study of vagueness the determinacy operator — as defined from in terms of borderlineness as above — is usually assumed to be a normal modal operator, and is plausibly a necessity operator in our sense.[18] But most people theorizing about vagueness in this way, at least implicitly, reject the idea that metaphysical necessity is broader than determinacy. In particular the following schema — a consequence of the claim that metaphysical necessity is the broadest necessity — is inconsistent with some commonly held assumptions:

If it's metaphysically necessary that A then it's determinate that A

The assumption in question is the appealing idea (explicitly accepted by many theorists) that the vague metaphysically supervenes on the precise: whether Harry is bald supervenes on how many hairs he has, whether a pile is a heap on how many grains constitute it, and so forth. One might think, as before, that all truths supervene on the facts about the values of fields at particular space-time points, and these are not the sorts of facts that could be vague.

Suppose that it is borderline whether Harry is bald, and that there is some precise truth p (that Harry has exactly n hairs, say) that metaphysically entails the truth about Harry's baldness. Moreover, since it is not determinately false that Harry is bald, and by assumption p is determinately true it follows that it's not determinately false that: p and Harry is bald.[19] Since it is not determinately true that Harry is bald either, by parallel reasoning it follows that it's not determinately false that: p and Harry is not bald. But if metaphysical necessity was the broadest necessity then everything that is not determinately false would be metaphysically possible (by contraposing the above conditional schema above and applying the duality of necessity and possibility). Thus it's metaphysically possible that (p and Harry is bald) and metaphysically possible that (p and Harry is not bald), contradicting the assumption that whether Harry is bald supervenes on p.

Indeed there are lots of positions in the wider philosophical literature that entail that metaphysical modality is not the broadest operator. For example, some philosophers think there can be counterpossibles: false counterfactuals with metaphysically impossible antecedents (see, e.g., Nolan [31], Brogaard and Salerno [7]). In that setting metaphysical necessity would not be broader than counterfactual necessity, defined as $\blacksquare A := (\neg A \Box\!\!\rightarrow \bot)$.[20] Similarly, some philosophers think that metaphysical necessity doesn't abide by the S4 principle (Chandler [8], Salmon [36]). In other words, $\Box A$ does not imply $\Box\Box A$, and so metaphysical necessity is not broader than its iteration $\Box\Box$.

quence of the standard semantics for tense logic that takes a good deal of getting used to, and it has recently been challenged by Dorr and Goodman [11]. Dorr and Goodman have things to say about both of the sorts of arguments that I have given above. They reject the coherence of an actuality operator satisfying the usual axioms, and cast doubt on the idea that everything supervenes on the eternal. However I find the latter idea so attractive that I have nonetheless not been won over by their arguments (I briefly treat this issue in Bacon [3], footnote 16 and the surrounding text).

[18] There is, of course, an open question whether determinacy operators are linguistic necessities or propositional necessities. Many theorists, such as McGee [29] and Williamson [45], assume it is a linguistic necessity, although others do not (see Fine [15], Field [14], Bacon [4], and Bacon [2]).)

[19] This is the modal inference from $\Diamond A$ and $\Box B$ to $\Diamond(A \wedge B)$ which is easily derivable in any normal modal logic, and so in particular holds when $\Box$ is interpreted as 'determinately'.

[20] $\blacksquare$ is a necessity operator since $(\neg\top \Box\!\!\rightarrow \bot)$, is equivalent to the logical truth $\bot \Box\!\!\rightarrow \bot$, which is itself plausibly the same proposition as $\top$. Moreover if $\neg A \rightarrow \bot$ was a counterpossible that just means that A is metaphysically necessary but not counterfactually necessary.

The idea that metaphysical necessity is the broadest necessity is thus not the uncontentious assumption that it is sometimes taken to be. Among other things, it takes a stand on the combination of views that maintain that (i) some propositions are temporarily true/borderline and (ii) the supervenience of all truths on the eternal/precise. It also excludes by fiat other open questions about the logic of metaphysical necessity and counterfactuals.

Our objection to the thesis that metaphysical necessity is the broadest necessity is not that there is some broader notion of necessity that subsumes it, for neither determinacy nor eternality are broader than metaphysically necessity. There are plenty of propositions that are always true or determinately true that are not metaphysically necessary. These operators are thus incomparable. This leaves it very much open whether there is a necessity operator broader than all three, or nothing broader than them.

2 Propositional Identity

The question of broadness is quite intimately connected with questions concerning propositional fineness of grain. If, for example, there is a proposition, A, distinct from but necessarily equivalent to $\top$, for some candidate notion of necessity $\Box$, then one can always find another necessity operator, $\Box'$, that doesn't apply to A. Since A and $\top$ are distinct, being identical to $\top$ applies to $\top$ but not A, so we can let $\Box'$ be the operator 'being identical to $\top$'. Since $\Box'\top$ just means $\top$ is identical to $\top$, and this fact is plausibly necessary for any weak necessity, $\Box'$ is a necessity operator. Moreover, $\Box$ is not broader than $\Box'$ since the material conditional $\Box A \to \Box' A$ is false, yet at minimum, the truth of this conditional is required for $\Box$ to be broader than $\Box'$ (substituting $\Box_3$ for truth in our definition of broadness). We may draw a general moral as follows: for any notion of necessity $\Box$, if propositions are more fine-grained that $\Box$-necessary equivalence, $\Box$ is not the broadest necessity operator.

We can make this argument (and the logic of fineness of grain more generally) explicit by introducing a binary propositional connective, $A = B$, into the propositional calculus that expresses the idea that A is the same proposition as B. (The propositional identity connective must be distinguished sharply from the identity relation between individuals denoted by the same symbol; since the latter will not feature prominently in this paper no confusion should arise.)

We can add axioms to the propositional calculus that tell us how propositional identity should behave. Taking the first-order theory of identity as our guide, it's natural to start with versions of the law of self-identity and Leibniz's law:

Identity: $A = A$

Substitution: $A = B \to (\phi \to \phi[A/B])$

Whatever your views about fineness of grain, every proposition is identical to itself, and if two propositions are the same, then either they are both ϕ or neither are, so they are intersubstitutable in all contexts ϕ.

The further assumption of Booleanism can be imposed via the following rule of inference:

Rule of Equivalence: If $\vdash A \leftrightarrow B$ then $\vdash A = B$

Every Boolean identity — say $A \wedge B = B \wedge A$ — can be proved from this rule, since it is possible to prove the corresponding equivalence given the propositional calculus — in this case $A \wedge B \leftrightarrow B \wedge A$. However the rule of equivalence is strictly stronger than Booleanism: it allows us to prove the identity of propositions that can be proved equivalent given the propositional calculus with the laws of IDENTITY and SUBSTITUTION. For example, since one can prove that $A = B \leftrightarrow B = A$ (exactly as one proves the symmetry of first-order identity) one can prove the identity $(A = B) = (B = A)$ even though it is not a Boolean identity.

Suszko [43] and Cresswell [9] have explored a broad notion of logical necessity that can be defined in terms of the identity connective.[21] A proposition is necessary, on this conception if it is identical to a logical truth. In general we shall write LA as short for the formula:

Definition of L: $A = \top$

A number of nice properties can be proven of this operator, as follows. (Rather than giving by and large uninformative axiomatic proofs, I give informal explanations of how to produce them in the following propositions.)

Proposition 2.1. *L is a normal modal operator whose logic includes* $\mathsf{S4}$.

Proof. To show that L is normal we must show that L obeys necessitation and the K axiom. For necessitation, suppose that we have a proof of A in our system. So we may also prove $A \leftrightarrow \top$ and by the Rule of Equivalence $A = \top$.

To show K it suffices to get $B = \top$ from $(A \to B) = \top$ and $A = \top$. An instance of substitution states: $(A = \top) \to (((A \to B) = \top) \to ((\top \to B) = \top))$, substituting $\top$ for A in $A \to B$. Applying modus ponens twice to both our assumptions we get that $(\top \to B) = \top$, and since $(\top \to B) = B$, by the Rule of Equivalence, we can apply Substitution again to get $B = \top$ as required.

T amounts to the claim that $A = \top \to A$. An instance of Substitution is $A = \top \to (\top \to A)$ which yields the desired conclusion by the propositional calculus.

$\mathsf{4}$ amounts to the claim $A = \top \to (A = \top) = \top$. Note that we can prove $(\top = \top) = \top$ by Identity and the Rule of Equivalence. Assuming $A = \top$ Substitution allows us to substitute the first $\top$ for A, getting us $(A = \top) = \top$ as desired. $\square$

It's also worth noting that the assumption of Booleanism, as encoded in the Rule of Equivalence, allows us to prove that propositional identity is just a kind of necessary equivalence. This is a distinctive consequence of Booleanism: such a result clearly cannot hold on a structured account of propositions, for example.

Proposition 2.2. *$A = B$ is provably equivalent to $L(A \leftrightarrow B)$*

Proof. To show the left to right direction assume $A = B$. By necessitation for L we can show that $L(A \leftrightarrow A)$. So by our assumption and Substitution we may conclude $L(A \leftrightarrow B)$ substituting the second A for B.

To show the right to left direction assume $(A \leftrightarrow B) = \top$. Now $(A \leftrightarrow B) \to A$ is logically equivalent to $(A \leftrightarrow B) \to B$, given the propositional calculus, so the Rule of Equivalence ensures that $((A \leftrightarrow B) \to A) = ((A \leftrightarrow B) \to B)$. Applying Substitution we can conclude $(\top \to A) = (\top \to B)$, substituting $\top$ for $A \leftrightarrow B$, since they are identical by our assumption. And finally, since we can prove from the Rule of Equivalence that $(\top \to A) = A$ and $(\top \to B) = B$ it follows, applying Substitution twice, that $A = B$ as required. $\square$

[21] The differences between these authors mainly consists in whether the RULE OF EQUIVALENCE is accepted.

Note, of course, that given the Rule of Equivalence we can conclude from proposition 2.2 that $(A = B) = L(A \leftrightarrow B)$. Thus identities are identical to necessary equivalences.

From our notion of L-necessity we can define other useful notions. L-possibility can be introduced as the dual of L: $MA := \neg L \neg A$. Logical entailment is naturally expressed in terms of the identity connective: A entails B if and only if $A \wedge B = A$. Given proposition 2.2, this is equivalent to $L((A \wedge B) \leftrightarrow B)$ which, by Booleanism, is equivalent to $L(A \rightarrow B)$. Thus entailment is similarly equivalent to a kind of strict implication.

Note that L-necessity materially implies every other weak necessity operator: if $\Box$ applies to $\top$ and A is L-necessary (i.e. A is identical to $\top$) then $\Box$ applies to A as well, by Substitution.[22] This feature is a direct result of defining L-necessity out of propositional identity, and we did not need to appeal to Booleanism to prove it. This fact also strongly suggests that the question we raised at the beginning about the existence of a broadest necessity operator can be settled positively. This suspicion is confirmed by the following theorem:

Theorem 2.3. *L is (i) a necessity operator and (ii) is at least as broad as any other necessity operator.*

Proof. To show (i) we must show that $\Box L\top$ is true whenever $\Box$ is a weak necessity operator. That is, we must prove the schema $\Box\top \rightarrow \Box L\top$, schematic for the operator expression $\Box$. We can prove $LL\top$ from necessitation which, expanding the definition of L, says $L\top = \top$. Substitution then entails that $\Box\top \rightarrow \Box L\top$ as required, substituting $\top$ for $L\top$ in the theorem $L\top \rightarrow L\top$.

For (ii) we must show that if $\Box$ is another necessity operator then L is broader than $\Box$. I.e. if $\Box'$ is a necessity operator, then $\Box'(LA \rightarrow \Box A)$. We shall in fact show the stronger result that this holds whenever $\Box'$ is a weak necessity operator. (Note that the idea that $\Box$ is a necessity operator is expressed as a schema $\Box''\top \rightarrow \Box''\Box\top$ (schematic in $\Box''$), so we are effectively proving that $\Box'\top \rightarrow \Box'(LA \rightarrow \Box A)$ follows from this schema. In fact, we will only need the instance of the schema $L\top \rightarrow L\Box\top$.)

Let $\Box'$ be some weak necessity operator. Note that we can get $\Box\top \rightarrow ((A = \top) \rightarrow \Box A)$ by rearranging Substitution, and since this claim is provable it is logically equivalent and thus identical to $\top$. Since $\Box'\top$ we can conclude $\Box'(\Box\top \rightarrow ((A = \top) \rightarrow \Box\top))$ by substitution.

Since $\Box$ is a necessity operator we know that $L\Box\top$ (since L is a weak necessity operator), i.e. $\Box\top = \top$. So by Substitution again we get $\Box'(\top \rightarrow ((A = \top) \rightarrow \Box\top))$, and finally $\Box'(LA \rightarrow \Box\top))$ by substituting Boolean equivalents and introducing L via its definition. $\Box$

3 Higher-Order Logic

In the last section we demonstrated that there is a broadest necessity on the assumption that there is a binary 'propositional identity' connective satisfying Identity, Substitution and the Rule of Equivalence. However someone skeptical of the idea of a broadest kind of necessity might also be skeptical of the notion of propositional identity, insofar as this is understood as a very broad kind of necessary equivalence (see proposition 2.2). It would be

[22] This fact was noted in Bacon [4] under the 'Broadness Theorem'. There the fact that the material conditional $LA \rightarrow \Box A$ could be proved, for every weak necessity operator, was taken to be sufficient for L to be broader than $\Box$. However since here we have adopted a more demanding notion of broadness a slightly more intricate argument is required. (As noted there, to prove the material conditional one does not need to assume Booleanism.)

nice if we could prove both existence of a broadest necessity and the propositional identity connective from some more general theory, without having to assume one to get the other.

Even if there exist connectives corresponding to the broadest necessity and propositional identity, these would be highly intensional connectives. It would be similarly nice to be able to give these connectives some sort of reductive analysis, in non-intensional terms, as some philosophers have attempted in the case of metaphysical modality (e.g. Lewis [27], Armstrong [1]). In the case of metaphysical modality most of these projects are extremely contentious. By contrast I believe that we have a route to something like a reductive analysis of the broadest necessity in broadly logical terms: we can define them terms of the higher-order universal quantifiers and truth functional connectives. This is surprising in light of the fact that no intensional notions can be straightforwardly defined from the first or second-order quantifiers and truth functional connectives.

To do this, however, will require us to theorize in the language of *higher-order logic*: a very general framework for reasoning about connectives and other expressions that do not take the position of a singular term. For example, in section 1 and 2 we frequently had to reason about *all* operators of a certain sort (see the definitions of broadness and necessity operator). In the background we achieved this by means of infinite schemata, but a lot of this reasoning would be made a lot simpler if we were able to *quantify* into the position that these operators occupy, and higher-order logic is exactly the sort of framework in which this sort of thing can be done.

First-order quantifiers allow one to quantify into the position that a singular term occupies. For example:

1. $\Box Fa$

2. $\exists x \Box Fx$

in 2 a bound variable is taking the place that the singular term a takes in 1 (and this inference is, incidentally, valid). In higher-order logic one can quantify into the position of any expression whatsoever. So for example the inference from 1 to $2'$ expresses the idea that there is some property a necessarily has by quantifying into the position the predicate F occupies:

$2'$. $\exists X \Box Xa$

Following Frege, we shall call the things in the range of the variable X *concepts*. Similarly, we can quantify into the position that the operator $\Box$ occupies to conclude that some operator applies to the proposition that Fa:

$2''$. $\exists X\, XFa$

Another ingredient of the version of higher-order logic we shall be employing here is the use of λ expressions to create further operators, predicates and other expressions using expressions we already have. For example, suppose we wish to express the fact that the complex predicate *being necessarily* F instantiates the higher-order concept of *applying to* a. We would first need a predicate in our language expressing being necessarily F (the obvious candidates don't work: $\Box Fx$ is an open sentence, not a predicate, and $\Box F$ isn't a well-formed expression since operators cannot take predicates as arguments). This is achieved by the use of a lambda expression $\lambda x\, \Box Fx$. The concept of applying to a can similarly be formalized using a lambda expression $\lambda X\, Xa$. Thus we may write $(\lambda X\, Xa)\lambda x\, \Box Fx$.

λ expressions are governed by two rules that intuitively say that lambda-abstraction and application are inverses of each other. The β rule says that a term of the form $(\lambda x\,\phi)\alpha$ can always be substituted *salve veritate* for $\phi[\alpha/x]$ provided α doesn't contain any variables that get bound upon substitution. Thus, for example, by β-reduction $(\lambda X\,Xa)\lambda x\,\Box Fx$ is equivalent to $\Box Fa$ (exercise). In a sense the β rule says that application is a left-inverse of lambda-abstraction: if I abstract x from ϕ (i.e. $\lambda x\phi$) and then apply the result to x (so $(\lambda x\phi)x$) I get something equivalent to what I started with: ϕ. The η rule intuitively says that application is a right inverse of lambda abstraction: if I apply a predicate ϕ to a variable (i.e. ϕx) and lambda out the variable (i.e. $\lambda x\phi x$) I get something equivalent to what I started with. Formally this means that ϕ is the same, and thus can always be substituted for $\lambda x\phi x$ (e.g. '... is a dog' and '... is an x such that x is a dog' are equivalent).[23]

A single variable cannot be used to take the position of expressions of more than one type: every variable is associated with a unique type, usually indicated by a subscript x_σ. In practice we will keep type subscripts tacit if they can be inferred from the context: in 2 x has type e, in $2'$ and $2''$, X has types $e \to t$ and $t \to t$ respectively. The sorts of types an expression can take on include the type of a sentence, singular term, predicate, operator and so on. A complete specification of the possible types of expressions in our language may be prescribed as follows. There are two basic types, e and t, standing for *entity* and *truth evaluable*, representing the types of sentences and singular terms respectively. If σ and τ are types then there is also a type $\sigma \to \tau$ representing expressions that take an expression of type σ as argument and produce an expression of type τ when concatenated. Since an operator takes a sentence as argument and results in another sentence its type is $t \to t$. Since a predicate takes a singular term and gives a sentence back its type is $e \to t$. If you supply something of type $e \to e \to t$ with two singular terms in succession you get a sentence, so this is naturally the type of a binary relation, and for similar reasons $t \to t \to t$ is the type of a binary connective. (In the preceding, and elsewhere, we associate brackets in types to the right, so that $e \to e \to t$ stands for $e \to (e \to t)$; we also follow the usual convention of dropping outermost brackets.)

In general every constant and variable of the language will be assigned a unique type. If M and N are expression of type $\sigma \to \tau$ and σ respectively then MN has type τ. If x has type σ and M has type τ then $\lambda x\,M$ has type $\sigma \to \tau$. The convention is to use upper-case variables when the variable is being applied to something, and lower-case variables when something is being applied to it. The convention has to break, however, when the same variable is applied and has something applied to it in the same term.

Higher-order logic is a particular family of typed languages that contain a binary connective $\to$ (which, recall, has type $t \to (t \to t)$) and for each type σ a quantifier $\forall_\sigma$ of type $(\sigma \to t) \to t$ that generalizes over things of type σ. If F is a predicate of type σ things (so F has type $\sigma \to t$) then $\forall_\sigma F$ intuitively says that F applies to everything of type σ. Note that on this interpretation $\forall_t X\,X$ defines the contradictory proposition $\bot$, and from $\to$ and $\bot$ all the truth functional connectives can be defined. The existential quantifier $\exists_\sigma$ can then be defined by duality.[24] Quantifiers in this setting do not bind variables. If we want to bind a free variable in an open formula, such as $\Box Fx$, we first λ abstract to a get a predicate and then apply the quantifier, as in the following example: $\exists_e \lambda x\,\Box Fx$. In such

[23]It is not often noted, but the principle of α equivalence, that allows one to re-letter bound variables, can be derived from the $\beta\eta$ rules (which is why we have not included it). A term of the from $\lambda x\,\phi$ is η-equivalent to $\lambda y \lambda x\,\phi y$ which by applying β reduction to the subterm $(\lambda x\,\phi)y$ is equivalent to $\lambda y\,\phi[y/x]$. This gives us α-equivalence since any relettering of bound variables in a term will amount to a relettering of a subterm of the form $\lambda x\phi$ (λ is the only variable binder in the language).

[24]More precisely, $\exists_\sigma := \lambda X\,\neg\forall_\sigma \lambda x\,\neg Xa$.

cases we suppress the λ yielding the more familiar formula 2: $\exists x \Box Fx$ (here we also suppress the type subscript on the quantifier when no confusion arises). The reader may refer to the appendix for more precise definitions if they wish.

It should be noted that we have so far been talking informally about 'propositions', 'operators', 'concepts', 'quantifiers' and so on; grammatically these are predicates taking singular terms as arguments. As such this is an inherently sloppy way to paraphrase quantification into positions occupied by non-singular expressions (see, for example, the famous 'concept horse' problem; Frege [17]). If we were truly to express in English what we achieve with a quantified formula of higher-order logic, we would have to take liberties with English: 'because John walks, he somethings', although it is not obviously unintelligible, is questionable English. We entrust it to the reader to translate our sloppy singular talk of operators and concepts into the relevant sort of quantification in higher-orderese. We will not attempt to give a full defense of the intelligibility of higher-order quantification here; for this we refer the reader to Frege [17], Prior [33] chapter 3 and Williamson [46].

Since most of our earlier reasoning only called for the use of quantification into the position an operator occupies, it is worth asking why we have decided to conduct our investigation in full higher-order logic with λ expressions and not merely the fragment that allows quantification into operator position and doesn't use λ expressions. One reason is that many true principles about operators fall out of the full theory which would otherwise have to be imposed in a piecemeal way. For example, at one point we argued that if L was the broadest operator it has to be broader than LL. But to make this reasoning precise we would need a principle that guaranteed that if an operator existed so does its iterations. More generally we want to ensure that the domain of the operator quantifiers is *closed under composition*. This is a simple consequence of higher-order logic, for any operators N and O we have the operator $\lambda x\, NOx$ (which we can instantiate with our universal quantifiers and from which we can infer existential generalizations). More generally we have a composition operator of the comparatively high type $(t \to t) \to (t \to t) \to t \to t$ defined as $\lambda X \lambda Y \lambda x\, XYx$, which takes two operators as input and outputs their composition. These sorts of points demonstrate the utility of including λ expressions and the portion of type hierarchy the lies beyond the operator fragment.

Note that with the power of higher-order logic at our disposal, key concepts introduced in the opening section can now be given precise definitions in terms of quantification into operator position. For example:

1. The claim that $\Box$ is a necessity can be expressed with the formula $\forall X(X\top \to X\Box\top)$. We shall abbreviate this $Nec(\Box)$.

2. The claim that $\Box_1$ is broader than $\Box_2$ can be expressed with the formula $\forall X \forall y(Nec(X) \to X(\Box_1 y \to \Box_2 y)$, where X has type $t \to t$ and y type t.

Moreover, it is exceedingly simple to define a weak necessity operator that materially implies every other weak necessity. We shall call this operator L, reusing our notation from section 2.

> Broad Necessity: A is broadly necessary if and only if every weak necessity operator applies to A.
>
> $LA := \forall X(X\top \to XA)$.[25]

[25] The operator L itself is defined as follows: $L := \lambda Y\, \forall X(X\top \to XY)$.

If a proposition is broadly necessary, then it is metaphysically necessary, *a priori* true, always true, determinately true, and so on, since each of these are weak necessities. More generally, if every weak necessity operator applies to A, and $\Box$ is a weak necessity operator then $\Box A$. So the material implication $LA \to \Box A$ holds whenever $\Box$ is a weak necessity operator. Showing that L is in fact broader than $\Box$ requires showing that the strict implication $\Box'(LA \to \Box A)$ is true for every necessity operator $\Box'$ — something we shall do shortly. For now note that our definition of L can be carried out entirely in logical vocabulary, so that if it can be shown that L is the broadest necessity we will have succeeded in giving it a seemingly reductive definition, in pure logic no less.

It is also possible to define a binary connective that satisfies both self-identity and substitution. Again we will reuse our terminology from section 2 and denote it with the symbol = (this terminological overloading will receive a proper justification shortly.)

> Leibniz Equivalence: A is Leibniz equivalent to B if and only if every operator applying to A applies to B.
>
> $A = B := \forall X(XA \to XB)$[26]

$A = A$ just amounts to the obvious truth $\forall X(XA \to XA)$. Moreover if A and B are Leibniz equivalent and ϕ is any term of type t then we can infer $\phi \to \phi[A/B]$. For A and B are Leibniz equivalent that every operator applying to A applies to B, so in particular if the operator $\lambda X\, \phi[X/B]$ applies to A it applies to B too. Since Leibniz equivalence satisfies the two characteristic axioms for identity we have used the symbol =, and we shall often call Leibniz equivalent things simply *identical*.

A variant definition of Leibniz equivalence replaces the conditional with a biconditional: $\forall X(XA \leftrightarrow XB)$. That these definitions are equivalent can be demonstrating by showing that the first definition of Leibniz equivalence is symmetric. For if every operator apply to A applies to B then for any operator, X, of type $t \to t$ there is a the composite operator $\lambda x\, \neg Xx$ (where x is of type t). We know that if this operator applies to A it applies to B — i.e. $\neg XA \to \neg XB$ — so by contraposition we know that if $XB \to XA$ as required. It is worth noting that according to our definitions being broadly necessary is the same as being Leibniz equivalent to $\top$, since LA and $\top = A$ expand to the same thing. Given the symmetry of = this means that LA and $A = \top$ are equivalent — mirroring the characteristic relationship between L and = we explored in the last section.

A conspicuous objection to the foregoing runs as follows: since the higher-order quantifiers used to define L (and =) range over intensional entities, the quantifiers must themselves be somehow intensional, meaning that we have not succeeded in defining an intensional notion from non-intensional notions after all. But this sort of reasoning seems in general to be suspicious: note that even if there are intensional first-order objects — entities of type e such as universals, that unlike sets, are not individuated by their extensions — few are inclined to consider the first-order quantifiers intensional. Nonetheless, many philosophers have found the idea that one can define intensional notions from extensional notions to be mysterious: it bears a close resemblence to contentious theses such as the idea that it is possible to reduce the modal to the non-modal, or the thought that it is possible to derive an *ought* from an *is*. It will thus be an instructive exercise to examine how this happens in a bit more depth.

An intensional operator is an operator within which materially equivalent propositions are not intersubstitutable *salve veritate*. More generally, an *intensional context* is one in

[26]The connective itself is defined: $\approx := \lambda Y \lambda Z\, \forall X(XY \to XZ)$.

which coextensive concepts cannot be intersubstituted *salve veritate*; if a context is not intensional say that it is *extensional*. Someone sympathetic to the foregoing line of thought might reasonably conjecture that it is possible to prove that anything definable from extensional predicates using only λ must also be extensional.

It is possible, in higher-order logic, to make this conjecture precise by formulating definitions of intensionality and extensionality. For simplicity we shall restrict our investigation to *relational types*. These are specified as follows: t is a relational type and $\sigma \to \tau$ is a relational type whenever τ is a relational type and σ any type. Intuitively a relational type will have the form $\sigma_1 \to \sigma_2 \to ...\sigma_n \to t$, and will thus represent an n-ary relation between things of types $\sigma_1...\sigma_n$. For each relational type σ we now define an equivalence relation $\sim_\sigma$ of coextensiveness as follows.

Coextensiveness

- If ϕ and ψ have type t then $\phi \sim_t \psi$ means $\phi \leftrightarrow \psi$.
- If τ is a relational type and ϕ and ψ have type $\sigma \to \tau$ then $\phi \sim_{\sigma\to\tau} \psi$ means $\forall_\sigma x(\phi x \sim_\tau \psi x)$.

For example, the claim that *renate* is coextensive with *cordate* amounts to the claim that, for every x, x is a renate if and only if x is a cordate. Two binary relations, for instance, *loves* and *likes*, are coextensive if for every x and y, x loves y if and only if x likes y, and so on.

A context is extensional if coextensive things can be substituted *salve veritate*:

Extensional Contexts: If $\sigma_1...\sigma_n$ are relational types and ϕ is of type $\sigma_1 \to ...\sigma_n \to t$ then ϕ is is *extensional* if and only if $\forall x_1...x_n y_1...y_n(x_1 \sim_{\sigma_1} y_1 \wedge ... \wedge x_n \sim_{\sigma_n} y_n \to \phi x_1...x_n \to \phi y_1...y_n)$.

Intensional Contexts: ϕ of the same type is *intensional* if it is not extensional.

Thus, for example, a unary operator θ is extensional if $\forall xy((x \leftrightarrow y) \to \theta x \to \theta y)$.[27]

Our conjecture that it is impossible to define an intensional context from extensional ones in the λ-calculus is now precise. But it is now possible to see that it is in fact false:

Observation: If there is a least one intensional operator then L is an intensional operator.

If there is at least one intensional operator, X (metaphysical necessity, say) then there are two materially equivalent propositions x and y such that Xx and $\neg Xy$. That is to say that we have materially equivalent propositions that aren't Leibniz equivalent. Consider the case in which they are both true: since they are distinct at least one of them — call it z — is distinct from $\top$ and so we have a truth that is not L-necessary. So $L\top$ and $\neg Lz$ even though $\top$ and z are materially equivalent. If they are both false, then since $\neg x$ and $\neg y$ are distinct, at least one of them is distinct from $\top$ and so by similar reasoning we have a z materially equivalent to $\top$ which is not L-necessary, so L is intensional in that case too. In other words:

[27] On this conception it can be contingent whether a relation or property is extensional. For example, the actuality operator @ counts as extensional since material equivalents are in fact substitutable within the scope of @, but it wouldn't have been extensional had things been any other way. There is thus a more demanding notion of being *broadly necessarily extensional* which could also be considered in this context; the result discussed below that L can be defined from extensional notions also shows that L can be defined from notions that are broadly necessarily extensional.

the only way in which L could be extensional would be if the Fregean view were right and there were only two propositions (in which case all operators would be extensional).

On the other hand we also have:

Observation: $\forall_\sigma$ and $\to$ are extensional.

Clearly $\to$ is extensional because it is truth functional. Moreover, suppose that F and G are coextensive predicates of type $\sigma \to t$. If F and G are coextensive — $\forall_\sigma x(Fx \leftrightarrow Gx)$ — then if $\forall_\sigma F$ it follows that $\forall_\sigma G$ by standard quantificational reasoning. Thus $\forall_\sigma$ is an extensional context. It follows, given the assumption that there's at least one intensional operator, that L is intensional even though it is definable in the typed λ-calculus from extensional primitives. Indeed, once this is noted we quickly find many intensional notions that can be defined from extensional notions — propositional identity, which we defined as Leibniz equivalence, is another such example.

The thesis we have established is that it is possible to define intensional notions from extensional notions in the simply typed lambda calculus. One might therefore wonder if the lambda calculus itself could be a source of intensionality. It's not clearly meaningful to ask whether the variable binding expression λ itself is an intensional or extensional notion, since it is not really a relation or property of any type. However it should be noted that one can define intensional notions in the λ calculus alone: application, $\lambda X \lambda y\, Xy$, is one such example, where X has type $t \to t$ and y type t. For an intensional operator M might apply to p and not q even when p and q are coextensive (i.e. materially equivalent). In which case we have (i) M is coextensive with M (trivial), (ii) p coextensive with q but (iii) M *applies* to p while it does not *apply* to q. So by definition application is not extensional. This is just another instance of the emergence of intensionality. (Since the λ calculus is a source of intensionality in itself, a natural question in the vicinity here is whether there is an extensional basis of combinators for the lambda calculus. The K combinator is extensional but the S combinator is not, so the textbook basis is not extensional; indeed it is easy to prove that there can be no extensional basis since applying an extensional combinator to something always results in an extensional output.)

Of course, one might simply reject our definition of extensionality and insist that the higher-order quantifiers used in our definition *must* count as intensional simply in virtue of the fact that one is able to define intensional notions from them. By these lights the very project of giving an analysis of an intensional notion in non-intensional terms is misguided; we can know that it will fail before we even try. Analogous reasoning can be used to show that if David Lewis is correct — if there are many concrete worlds spatio-temporally disconnected from our own, and modal notions can be defined in terms of first-order quantification over them — then first-order quantification must be deemed intensional too, and so a reduction to non-intensional notions must have failed. I have little to say about this sort of attitude towards reductions, except that I anticipate that few apart from the most die-hard anti-reductionist will be persuaded. If there are problems with Lewis's account of metaphysical modality, it was not that the project was incoherent from the get go. My own view, at any rate, is that if such reductions are possible at all (and I'm inclined to think that they are), then our definition of broad necessity is one of them.

Making the informal arguments we gave above precise requires we be a bit more explicit about the sort of system we are working in. To do so we shall describe a theory within which this reasoning can be performed. A theory is just a set of terms of the language of type t. An axiom (such as UI below) is to be understood as a member of the theory, and a rule (such as Gen below) is to be understood as saying that the conclusion of the rule is in the theory whenever the premises are. (When we later talk about rules being 'added' to theories, we will always mean that the theories are to be closed under the rules.) The theory below is called H.

PC All instances of propositional tautologies.

MP From A and $A \to B$ infer B

Gen From $A \to B$ infer $A \to \forall_\sigma x B$ when x does not occur free in A.

UI $\forall_\sigma x \phi \to \phi[t/x]$ (where t is a term of type σ)

$\beta\eta$ $A \leftrightarrow B$ whenever A and B are $\beta\eta$ equivalent terms of type t.

Terms are $\beta\eta$ equivalent if you can get one from the other using the β and η rules described earlier.

Leibniz equivalence is an abbreviation for a connective that holds between two things of type t. We could make the type of the arguments explicit by indicating the type with a subscript: $=_t$. Note however that the same idea could be used to define Leibniz equivalence holding between entities of any type. If a and b are terms of type σ we may write $a =_\sigma b$ to abbreviate $\forall X(Xa \to Xb)$ (where X has type $\sigma \to t$).

The theory H encodes the core of classical higher-order logic. However there are two further principles involving Leibniz equivalence that are part of my preferred version of higher-order logic. (Not all of the results will require these further principles; I shall always flag when a result does not.)

The first is a principle of functionality stating that if two functions X and Y of type $\sigma \to \tau$ output the same values for each of their arguments they are the same function:

Functionality $\forall_\sigma x(Xx =_\tau Yx) \to X =_{\sigma\to\tau} Y$

This is a principle telling us when things of functional type $\sigma \to \tau$ are Leibniz equivalent in terms of the Leibniz equivalence of their values.

We can also impose the Rule of Equivalence introduced in section 2, this time as a constraint on Leibniz equivalence. This is a constraint on Leibniz equivalence for terms of type t only:

Rule of Equivalence If $\vdash A \leftrightarrow B$ then $\vdash A =_t B$.

As before this principle encodes the assumption of Booleanism, as well as many other provable identities. Note that although the Rule of Equivalence imposes constraints on the structure of entities at type t, in conjunction with Functionality it implies analogues of these constraints for higher types. For example the Boolean operations can be lifted to predicates — predicate negation, $\neg'$, can be defined by the term $\lambda X \lambda y \neg Xy$, for instance — and these operations can be shown to satisfy the Boolean identities. For example, Booleanism at type t entails that $\forall x(Fx = \neg\neg Fx)$, which by β fiddling, is equivalent to $\forall x(Fx = (\neg'\neg' F)x)$. Applying Functionality yields $F = \neg'\neg' F$. Predicative versions of the other Boolean identities are proven in a similar manner. (Indeed, analogues of Booleanism hold at every relational type.)[28]

[28][REF] has pointed out to me that these results (and some of the results below) can be proven without the functionality axiom if we assume a strengthening of the Rule of Equivalence: if $\vdash Ax_1...x_n \leftrightarrow Bx_1...x_n$ then $\vdash A = B$. This can also be seen as a rule version of the axiom of Functionality.

Note that it is important to distinguish the Rule of Equivalence from what Suszko [44] calls the *Fregean Axiom*, which is a material conditional, and not a rule:[29]

The Fregean Axiom $(A \leftrightarrow B) \rightarrow A =_t B$

The Rule of Equivalence only guarantees the identity of logically equivalent things (things that can be proved to be logically equivalent in a given background system), and does not in general guarantee the identity of material equivalents. The Rule of Equivalence seems like the most natural way to weaken the Fregean axiom in a way that does not entail any of its undesirable consequences.

If the system includes Functionality or the Rule of Equivalence or both we shall call it HF, HE or HFE respectively. (Note that technically speaking the Rule of Equivalence is a closure condition on a theory, so that it adds more theorems when combined with HF than with H since we can prove more things materially equivalent given functionality. This same point applies to the rules of Modus Ponens and Gen.)

From H we can give precise versions of the informal arguments that $=$ satisfies Identity and Substitution and that L materially implies every weak necessity operator we outlined earlier. It can also be shown that L satisfies $\mathsf{S4}$. Since this argument is relatively simple and mirrors arguments we have presented already I shall only look at some representative examples. The rule of necessitation for L follows straightforwardly from the rule of equivalence is it did in section 2. Applying necessitation twice yields $LL\top$, so we know that the operator $\lambda x\, LLx$ is a weak necessity operator. If every weak necessity operator applies to A — A is L-necessary — then in particular $\lambda x\, LLx$ applies to A, using the quantificational principle UI. Thus $LA \rightarrow (\lambda x\, LLx)A$, and the $\mathsf{S4}$ follows by β-conversion. The K and T principles also follow by similar arguments to ones we have considered earlier. For example, for any B the operator $\lambda X(L(X \rightarrow B) \rightarrow LB)$ can be shown to be a necessity operator using the Rule of Equivalence and the derived axiom of Substitution.[30] Thus if every weak necessity operator applies to A — i.e. A is L-necessary — then this operator in particular applies to A so $L(A \rightarrow B) \rightarrow LB$ establishing K.

This should all be relatively familiar. In the higher-order setting we can also raise questions about the interaction of the quantifiers with broad necessity. The most famous of these is the Barcan formula (see Barcan [5]):

The Barcan Formula $\forall x LA \rightarrow L\forall x A$

The principle is most well known in the form where x is a singular variable ranging over things of type e, but here I intend the schema to be good for every well-formed instance involving variables of any type.

The Barcan formula may be proved in HFE with the help of the functionality principle. Modulo definitions the antecedent says: $\forall x(A = \top)$. By β-equivalence this is equivalent to $\forall x((\lambda y\, A)x = (\lambda y\, \top)x)$. So by functionality $\lambda y\, A = \lambda y\, \top$. So by Substitution $\forall \lambda x\, A(x) = \forall \lambda x\, \top$ (applying $\forall$ to both sides). Finally by quantificational logic we can prove that $\forall x\, \top \leftrightarrow \top$ so by the Rule of Equivalence we have $\forall x\, \top = \top$. Making that substitution gives us $\forall \lambda x\, A(x) = \top$, or equivalently $L\forall x A$, as required.

[29] Suszko does not state his principle in full-fledged higher order logic, and so his version takes the form of a schema. Without employing higher-order resources, like Leibniz equivalence, it amounts to the claim that all operators are extensional: $A \leftrightarrow B \rightarrow \phi \rightarrow \phi[A/B]$.

[30] The rule of equivalence proves $B = \top \rightarrow B$. An instance of Substitution says $B = (\top \rightarrow B) \rightarrow ((LB \rightarrow LB) = (LB \rightarrow LB)) \rightarrow ((L(\top \rightarrow B) \rightarrow LB) = (LB \rightarrow LB))$, so we may conclude $((L(\top \rightarrow B) \rightarrow LB) = (LB \rightarrow LB))$. Since $LB \rightarrow LB$, we may conclude $(L(\top \rightarrow B) \rightarrow LB)$ (by substitution again, substituting the whole formula $LB \rightarrow LB$ for the conclusion). This is the desired conclusion.

It is worth noting that the Barcan formula has a very different status from it's converse, which it is straightforwardly provable in H. The converse is thus unavoidable without modifying the logic of the quantifiers.[31]

The Barcan formula has many interesting upshots. One of these is the fact that the universal quantifier behaves like a greatest lower bound. Recall that A entails B, which we may write $A \leq B$, iff $A = A \wedge B$, and recall that this is equivalent to the strict implication $L(A \to B)$ (see the discussion under proposition 2.2). Applying necessitation and Gen to the following instance of UI, $\forall x Fx \to Fy$, we obtain $\forall y L(\forall x Fx \to Fy)$. This is equivalent to $\forall y(\forall x Fx \leq Fy)$ which is just to say that $\forall x Fx$ is a lower bound of the Fy. Now suppose that A is another lower bound: $\forall y(A \leq Fy)$. That's equivalent to $\forall y L(A \to Fy)$ and so by the Barcan formula and quantificational reasoning $L(A \to \forall y Fy)$. And this is equivalent to $A \leq \forall y Fy$. Since A was an arbitrary lower bound, $\forall y Fy$ must be the greatest lower bound.

The result of most importance for our purposes is that L is a necessity operator and is moreover broader than any other necessity operator. Note that in the context of higher-order logic we can state both of these ideas using single formulae, as opposed to schemata:

Proposition 3.1. *The following are theorems of* HE*:*

i. L is a necessity operator: $\forall X(X\top \to XL\top)$

ii. L is broader than every necessity operator: $\forall XY(Nec(X) \to Nec(Y) \to \forall xY(Lx \to Xx)))$.

The proof of this theorem is in fact just a rehashing of theorem 2.3, which was proved from the propositional calculus, Identity, Substitution, the Rule of Equivalence, and the equivalence of LA with $A = \top$ (which in that setting was guaranteed by a definition). Since each of these principles hold in the present setting, interpreting $=$ as Leibniz equivalence the result stands. Indeed all of the results 2.1, 2.2 and 2.3 proved there carry over to the present setting.

This almost completes our argument that L is the broadest necessity. However it should be noted that the conclusion that L is *the* broadest necessity is a little premature. For all we've said at this juncture there could be another operator, L', that is at least as broad as every operator. This would mean that L' was at least as broad as L, and vice versa, but it does not (yet) mean that they are the same. To show this we will need to make use of our assumption of Functionality introduced earlier.

Proposition 3.2 (Uniqueness of broad necessity (HFE))**.** *There is at most one broadest necessity operator.*

Proof. If L and L' are two maximally broad necessity operators, then they are at least as broad as each other.

That L is at least as broad as L' means that every necessity operator applies to $(Lx \to L'x)$, and since L is a necessity operator $L(Lx \to L'x)$. Since L' is at least as broad as L we can similarly conclude $L(L'x \to Lx)$. Since L is a normal modal operator (see theorem 2.1) we know that $L(Lx \leftrightarrow L'x)$. So by theorem 2.2 $Lx = Lx'$.

Generalizing we know that $\forall x(Lx = L'x)$, and finally applying functionality we can conclude $L = L'$. □

[31] $\forall x LA \to LA$ is an instance of UI (making the vacuous substitution of x for x). Applying Gen directly gives us $\forall x LA \to \forall x LA$ as required.

This theorem therefore justifies our terminological abuse of using L for both the thing we defined in section 2 from propositional identity, and here from Leibniz equivalence: since they are both broadest necessity operators, they are identical by proposition 3.2. The other notational abuse we made was to use the same symbol for Leibniz equivalence as propositional identity. This justified by the following similar theorem:[32]

Proposition 3.3 (Uniqueness of Identity (HFE))**.** *If $\approx$ and $\approx'$ are two binary relations on type σ such that:*

i. $\approx$ and $\approx'$ necessarily satisfy the law of self-identity: $Lx \approx x$ and $Lx \approx' x$

ii. $\approx$ and $\approx'$ necessarily obey substitution: $L(x \approx y \rightarrow \phi \rightarrow \phi[x/y])$, $L(x \approx' y \rightarrow \phi \rightarrow \phi[x/y])$

Then $\approx$ is Leibniz equivalent to $\approx'$: $\approx =_{\sigma\rightarrow\sigma\rightarrow t} \approx'$.

Proof. We first prove that $x \approx y \leftrightarrow x \approx' y$ from assumptions that are L-necessary. Since the assumptions are L-necessary we may conclude $L(x \approx y \leftrightarrow x \approx' y)$ (since L is a normal modal operator), and thus that $x \approx y = x \approx' y$ by theorem 2.2. We can then use functionality to show that $\approx = \approx'$: firstly since, for any x, $\forall y(\lambda z(x \approx z)y = \lambda z(x \approx' z)y)$ functionality tells us that for any x $\lambda z(x \approx z) = \lambda z(x \approx' z)$. By applying functionality again, and η conversion that gives us $\approx = \approx'$.

So it just suffices to prove $x \approx y \leftrightarrow x \approx' y$ from necessary assumptions. Since by assumption i) $x \approx' x$ and ii) $x \approx y \rightarrow x \approx' x \rightarrow x \approx' y$ we get $x \approx y \rightarrow x \approx' y$. A parallel argument establishes the other direction, so we have proven $x \approx y \leftrightarrow x \approx' y$ from assumptions that are L necessary. □

The theorem effectively shows that any relations necessarily satisfying Identity and Substitution are the same modulo Leibniz equivalence. In particular they must be the same as Leibniz equivalence, modulo Leibniz equivalence, since Leibniz equivalence satisfies both of the theorem's conditions.[33]

To illustrate this theorem, consider an alternative definition of identity in which a and b are identical iff every reflexive relation applies to a and b: $\forall X(\forall z\, Xzz \rightarrow Xab)$ (where a, b are of some type σ and X is a binary relation of type $\sigma \rightarrow \sigma \rightarrow t$). It is easy to prove, in HFE, that our alternative definition of identity satisfies Identity and Substitution and so by necessitation for L they do so of L-necessity. Therefore applying theorem 3.3, our alternative identity relation must be Leibniz equivalent to Leibniz equivalence.

However someone concerned that Leibniz equivalence doesn't express 'real identity' might nonetheless worry that this theorem doesn't really prove the uniqueness of identity. While the candidate identity relations may all be Leibniz equivalent to one another, the worry might go, we have no guarantee that they are really the same, and thus, in particular, we have no guarantee that Leibniz equivalence is *really* identical to *real* identity.

Luckily a more subtle application of the theorem can head that worry off as well. For suppose that there was a family of relations, $=^R_\sigma$, at each type σ, that *did* express real identity and that necessarily satisfies Identity and Substitution. We could then apply proposition 3.3 to show that $=_\sigma$ — Leibniz equivalence at type σ — and $=^R_\sigma$ are in fact Leibniz equivalent.

[32] The theorem below was proved using the Rule of Equivalence, however a version of it is provable without that assumption with a slightly more intricate proof.

[33] For this argument to make any sense one must take heed of the types of these identifications: every candidate identity relation between entities of type σ is Leibniz equivalent (at type $\sigma \rightarrow \sigma \rightarrow t$) to Leibniz equivalence at type σ.

That is to say, every concept applying to $=^\sigma_R$ applies to Leibniz equivalence $=^\sigma$. In particular, consider the concept: being *really identical* to real identity, or $\lambda X(X =^R_{\sigma\to\sigma\to t} =^R_\sigma)$. This property applies to $=^\sigma_R$, and so it applies to Leibniz equivalence. That is: $=_\sigma =^R_{\sigma\to\sigma\to t} =^\sigma_R)$. In other words, real identity is *really identical* to Leibniz equivalence.

4 The Logic of Broad Necessity

Let us turn our attention once again to the propositional modal language of the operator L. We saw in sections 1, 2 and 3, in different ways, that broad necessity satisfies the principles of the logic S4. However this fact that does not completely settle the question of what the modal logic of broad necessity is; since there could be other principles that are valid, S4 is merely a lower bound.

The notion of validity relevant for our purposes may be defined as follows: a closed term ϕ of type t in the language of higher-order logic is *logically valid* (or just *valid*) if it contains only logical vocabulary and the sentence $L\phi$ is true. A term is purely logical if it is constructed out of λ, variables, $\to$ and $\forall_\sigma$ only. An open term of type t is valid if its universal closure is. The restriction to sentences containing only logical vocabulary is so that we can focus on principles that are L-necessary but whose L-necessity does not depend on the interpretation of any non-logical predicate. The usual schemata of propositional modal logic, for example, can be understood as open sentences with variables replacing the schematic letters.[34]

An upper bound on the logic of broad necessity is the modal logic Ver, which is axiomatized by adding to the normal modal logic K the schema:

Ver $LA \leftrightarrow A$

There are no consistent modal logics that extend Ver (Hughes and Cresswell [22] p67) so it is indeed an upper bound. This would be the modal logic of broad necessity if the Fregean axiom were true: the principle that there are only two propositions (see previous section). For then every truth would be identical to $\top$ and since no falsehood can be identical to $\top$ a proposition is L-necessary if and only if it is true.

Since the Fregean axiom is plausibly false, Ver is not logically valid. Indeed we can employ an argument due to Joe Schiller Scroggs to get a bit further and show that S5 is an upper bound. S5, recall, is the result of adding the following axiom to S4:

5 $\neg LA \to L\neg A$

Or equivalently, by adding the Brouwerian axiom:

B $A \to LMA$

Scroggs shows that the logic of any proper extension of S5 is the logic of a frame with finitely many worlds. The theorems of this logic can be refuted by reflection on the truth that, for any n, it's metaphysically possible, and thus broadly possible, for there to have been exactly n stars, and the fact that these truths require the existence of infinitely many worlds. (In more detail, there will be object language validities in any proper extension of S5 for which

[34] It is worth comparing this with the notion of metaphysical universality adopted by Williamson [47]: in the language of higher-order logic it amounts to a sentence which is purely logical and true (as opposed to being purely logical and being broadly necessary). On Williamson's conception there could be metaphysically universal truths that aren't even metaphysically necessary.

one can transform these considerations into explicit counterexamples. See Scroggs [39] and Williamson [47] p111 for a discussion of this result.)

So this means the logic of broad necessity is somewhere between S4 and S5. But this leaves matters quite open – it leaves it open whether, for example, the McKinsey axiom, $\Box\Diamond A \to \Diamond\Box A$, or its converse, or any of a myriad of other principles, are valid.

It is worth mentioning that there are parallel questions about the logic of identity. One of these is the necessity of distinctness: that if two propositions are distinct they are necessarily distinct:

The Necessity of Distinctness: $A \neq B \to L(A \neq B)$.

Parallel questions concerning the necessity of distinctness arise at other types as well. In fact these questions are closely tied to the status of 5 and B. Substituting B for $\top$ in the above gives us 5, and there is a well-known argument getting us the necessity of distinctness from B (see Prior [34] pp.206-207).[35]

A tractable question that is open at this juncture concerns whether it is possible to prove any further modal principles about broad necessity from the system HFE. There might, for example, be a cunning argument in HFE for the S5 axiom or the McKinsey axiom or some other principle that we have just missed in our discussion in section 3. This question can in fact be answered negatively:

Theorem 4.1. *Consider the smallest set of terms, $\mathcal{L}_L$, in the language of higher-order logic that contains: (i) a (given) infinite set of constants of type t (ii) contains the sentences $\bot$, $L\phi$, $\phi \to \psi$ whenever it contains ϕ and ψ. Then for any $\phi \in \mathcal{L}_L$ the following are equivalent:*

There is a proof of ϕ (as a sentence of propositional modal logic) in S4

There is a proof of ϕ (as a sentence of higher-order logic) in HFE.

The theorem is proven by showing it is possible to construct, for any world in a transitive reflexive Kripke model, a model of HFE which makes exactly the same sentences true (this is done using Kripke logical relations; see Plotkin [32]). Since transitive reflexive Kripke frames are complete for S4 it follows that models of HFE are complete for S4. (It is worth noting, with regards to the preceding, that a model of HFE is just a model of propositional modal logic — via the interpretation it assigns to L — with a lot of redundant structure). So if there is a proof of something in HFE it is true in all models of HFE, and thus every transitive reflexive Kripke frame which finally means it must be provable in S4. The converse — that the theorems of S4 can be proven in HFE — we have shown already.

So although we can't prove any principles that go beyond S4 from our assumptions so far, a stronger logic may yet be valid. Here I have nothing conclusive to say, however there are some arguments against S5 that I take quite seriously that have lead me to believe that S5 is not the logic of broad necessity. I consider some arguments in favor of S5 in section 4.2, 4.3 and 4.4 as well.

[35] Failures of the necessity of distinctness is one way in which my definition of validity can come apart from Williamson's notion of metaphysical universality. For example there are models of HFE in which there are four propositions but, because two pairs of distinct propositions are possibly identical, it's possible that there are only two propositions. The claim that there are exactly four propositions is metaphysically universal, but not valid in my sense because it is not L-necessary. A natural conjecture would be that on the assumption of S5 every metaphysically universal sentence is valid (the converse is trivially true).

4.1 The Argument from Vagueness

So far we have concerned ourselves primarily with the logic of broad necessity without taking any stances on which propositions are broadly necessary beyond those expressed by theorems of HFE. Further plausible candidates include propositions expressed by conceptual truths, such as the truth that scarlet things are red, that everyone over the age of 40 is an adult, that people with no hairs are bald, and so on.

Note that by varying the age, the shade of color, and the number of hairs in the above examples we can generate sorites sequences of propositions that begin with broad necessities and segue into propositions that are clearly not broad necessities. Thus for example:

1. It is L-necessary that people with no hairs are bald.
2. It is not L-necessary that people with at most a million hairs are bald.

As noted, 1 is plausibly true because it's conceptually true that people with no hairs are bald, and 2 is true simply because it's not true that people with a million hairs are bald and L is factive. By a bit of logic there must be some n such it is L-necessary that people with at most n hairs are bald, but not L-necessary that people with at most $n+1$ hairs are bald. Despite this result (which must be accepted by anyone who accepts 1, 2 and classical logic) it is surely not a precise matter which this last broadly necessary proposition is. It is rather *borderline* where it lies. In particular for the critical n and (and the surrounding borderline cases):

3. It is *borderline* whether it is L-necessary that people with at most n hairs are bald.

We might formalize this ∇LA, where ∇B stands for the formula $\neg\Delta B \wedge \neg\Delta\neg B$: it is not determinate that B and not determinate that $\neg B$.

A logic of S5 would entail the impossibility of 3. For S5 guarantees both the conditionals $LA \to LLA$ and $\neg LA \to L\neg LA$. But since L is the broadest necessity we know that the conditionals $LLA \to \Delta LA$ and $L\neg LA \to \Delta\neg LA$ are true. Chaining these conditionals together gets us that $LA \to \Delta LA$ and $\neg LA \to \Delta\neg A$, which along with the instance of excluded middle, $LA \vee \neg LA$ entails $\Delta LA \vee \Delta\neg LA$ which is inconsistent with ∇LA.

4.2 The Argument from Kreisel's Principle

There are a couple of theoretical arguments in favor of S5 that I would like now to turn to. Both rely on technical assumptions that I do not think ultimately hold up to scrutiny. However the surrounding issues are helpful to think about and bring out clearly what things would have to look like in a non-S5 universe.

The first is a model theoretic argument. A model of higher-order logic assigns to each type σ a domain D_σ. At functional types $\sigma \to \tau$ the elements of $D_{\sigma\to\tau}$ determine functions from D_σ to D_τ. To evaluate terms of type t for truth we pick a subset T of D_t to represent the truths. Finally we select elements $cond \in D_{t\to t\to t}$ and $all \in D_{(\sigma\to t)\to t}$ to interpret the conditional and quantifiers, with the constraint that they interact with the set T in the right sort of way: we require that $cond(a)(b) \in T$ if and only if $a \notin T$ or $b \in T$, and we require that $all(f) \in T$ if and only if $f(a) \in T$ for every $a \in D_\sigma$.

In general two elements of domain $D_{\sigma\to\tau}$ can determine the same function from D_σ to D_τ, but when this never happens we call the model *functional*. If every function from D_σ to D_τ is determined by some element of $D_{\sigma\to\tau}$ then the model is called *full*. If furthermore the elements of D_t form a Boolean algebra, and the connectives and quantifiers express the corresponding Boolean operations (this implies that T forms an ultrafilter of D_t and that D_t has certain completeness properties) we call the model *Boolean*. If a model is Boolean and $|D_t| = 2$ we call the model extensional. Finally call a model *standard* if it is full, functional and extensional.

In first-order logic, if a sentence is true in every set theoretic model then it is true simpliciter. Since the set of all first-order truths is certainly consistent in first-order logic it follows by the completeness theorem for first-order logic that there is a model that makes all of those sentences true, so if a sentence is true in all such models it must be among the truths. This result holds even though there is no model in which the quantifiers have their intended range, for in every model the quantifiers are restricted to a set and so no model captures the unrestricted reading of the quantifiers. The reason is that there's nothing we can say in first-order logic that is true on the unrestricted reading of the quantifiers but is false on every reading in which they are restricted to a set. (See Kreisel [24].)

If we thought that standard models were the relevant analogue of first-order models for higher-order logic then this sort of argument would not be successful since we do not have an analogous completeness theorem for any recursively axiomatisable system relative to standard models. Shapiro [40] has investigated the result of adopting this principle as an independent assumption:[36]

> Kreisel's Principle: If a sentence of higher-order logic is true in all standard models then it is true.

It is easily verified that the theorems of $\mathsf{S5}$ for L are true in every standard model of higher-order logic: since there are only two propositions in an extensional model, every truth is identical to $\top$ and every falsehood identical to $\bot$ so if the proposition that $A = \top$ is true it is identical to $\top$ and if it's negation is true then it's negation is identical to $\top$. So Kreisel's principle entails the truth of the theorems of $\mathsf{S5}$.

As it stands, however, Kreisel's principle is not particularly plausible. It entails the Fregean axiom, since the Fregean axiom is true in all extensional models of higher-order logic. For similar reasons it entails that there are no intensional operators, and many other implausible theorems; in particular it entails that the logic of L is in fact Ver.

The contexts in which Kreisel's principle is usually applied are mathematical and in these cases it is not usually relevant whether or not there is intensionality. However if the principle is to be at all plausible we need to weaken the principle somehow. Consider the following candidate weakening:

> Weakened Kreisel's Principle: If a sentence of higher-order logic is true in all full, Boolean models then it is true.

This weakening no longer entails the Fregean axiom. It does not entail functionality either, although of course functionality may still be true.

This weaker version of Kreisel's principle also entails the $\mathsf{S5}$ principle:

Proposition 4.2. *$\neg LA \rightarrow L\neg LA$ is true in every full Boolean model of higher-order logic.*

This follows from the fact that full models contain the δ function discussed in the next section.

On the assumption of Booleanism, the restriction to Boolean models in Kreisel's principle does not seem to be an obvious source of contention. One might try to justify the restriction

[36]The version stated below is distinct from, but closely related to Shapiro's principle.

to full models as follows. When we quantify using a higher-order quantifier $\forall_{\sigma\to\tau}$ over concepts with a functional type, we are by definition quantifying over *all* such functions. After all, the only means we have to express the notion of every function of type $\sigma \to \tau$ is with the universal quantifier $\forall_{\sigma\to\tau}$. Full models of higher-order logic therefore give us a kind of miniature model of what things are like when you're quantifying over all functions, by contrast to non-full models in which you are only quantifying over some of the functions.

But the analogy between functions and concepts is prone to mislead. When we quantify into the position that an operator occupies, for example, we are not really quantifying over functions between propositions. Despite the sloppy way we talk the things that occupy types other than type e, they are not really well modeled by singular talk of propositions, functions, and the like. Even if the behavior of these miniature models can be taken to be a good guide to the world at large, it's far from obvious that every function from individuals to propositions corresponds to a property. For instance, if I apply a property F to an individual a one might expect the resulting proposition to be *about* that individual, yet an arbitrary function can map a to any proposition, whether about a or not.

In the context of the theory HFE presented above, there are several ways in which higher-order quantification into, say, predicate position can fail to be faithfully modeled by quantification over functions. For example, in the case of functions, bijections (one-to-one correspondences) are always invertible, in the sense that if f is a bijection there's a function g such that $g \circ f$ is the identity mapping. But there are models of HFE in which the analogue of this claim is not true (see appendix). On the one hand, there is a concept F (of predicate type, $e \to t$) such that for any distinct individuals a and b the proposition that a is F is distinct from the proposition that b is F, every proposition is identical to the proposition that a is F for some individual a. On the other, in this model, there is no concept G of type $t \to e$ such that $\lambda x\, GFx = \lambda x\, x$. (Interestingly, in these models all the one-to-one correspondences of type $e \to t$ are at best contingently one-to-one: this is because there are distinct things of type t that are only contingently distinct.[37])

We granted above that the behavior of 'small' set theoretic models of the type hierarchy might be a good guide to the world at large. But it is far from clear that this is true. It should be noted that Kreisel's principle, even in the weakened form above, is extremely strong and implies many surprising set theoretical theses, such as the existence of certain sorts of large cardinals. For example the claim that the domain of individuals has inaccessible size is something that can be stated with a sentence, A, in higher-order logic. Set theoretical assumptions guarantee that A is in fact true. Now suppose there were no *sets* of inaccessible size: then every set theoretic model makes A false even though it is in fact true, contradicting Kreisel's principle. Thus Kreisel's principle entails the existence of inaccessible cardinals (indeed it proves the existence of many other small large cardinals; see Shapiro [40]).

Apart from the set theoretical consequences, it implies non-obvious things about the relation between the sizes of type e and type t entities. For example if there are exactly as many propositions as individuals, as captured by the existence of a bijective concept $e \to t$, then Kreisel's principle entails, assuming functionality, that there are effectively no interesting differences, stateable in higher-order logic, between elements of higher-order types that can be obtained from one another by switching es and ts. For example, it entails operators (type $t \to t$) and predicates (type $e \to t$) are indistinguishable in this sense. This is because in full functional models, the behavior of higher types is completely determined

[37]The models can be generated using Kripke logical relations — the general technique is outlined in the appendix.

by the sizes of the base types.[38] These sorts of correlations are not provable in HFE.

It is worth asking what would happen if we were to relax the condition that the models be full in our statements of Kreisel's principle. In this case the principle has the same status as the first-order version of the principle. Assuming the set of truths contains all the theorems of HFE, then we can apply standard completeness arguments (due to Henkin [20]) — that can be derived in ordinary ZFC without any large cardinal assumptions — to show that there is a non-full model that set of truths. Moreover, due to the completeness theorem, the resulting principle will not entail anything that cannot already be proven in HFE.

4.3 The Argument from Church's δ Function

The argument from Kreisel's principle is inherently model theoretic. An alternative object language argument can be given in favor of S5 by assuming that a particular function belongs to the domain of operations on propositions. The function I have in mind for this purpose is known as Church's δ function.[39] In the present setting we may regard it as a four-place connective of type $t \to t \to t \to t \to t$.[40] In model theoretic terms it is the function that given arguments a, b, c and d from D_t outputs c if $a = b$ and d otherwise (it thus behaves a little like a programmers' conditional). In a full model it is always guaranteed that Church's δ function is a member of $D_{t\to t\to t\to t\to t}$. However there are models of higher-order logic that do not contain the δ function (the models generated by theorem 4.1 include many such examples). It follows, in particular, that the δ function cannot be defined from the λ terms, connectives, and higher-order quantifiers.

Although I described Church's δ function in model theoretic terms, its behavior can be captured purely in the object-language.

$\delta 1$ $A = B \to \delta ABCD = C$

$\delta 2$ $A \neq B \to \delta ABCD = D$

These principles will come out true in any model of higher-order logic in which δ is interpreted in the way described above.

Moreover, in any full model these principles will not only be true, but *valid*. We can capture this in the object language by the truth of the following two principles:

$L\delta 1$ $L(A = B \to \delta ABCD = C)$

$L\delta 2$ $L(A \neq B \to \delta ABCD = D)$

The the above strengthening of δ_1 and δ_2 is sufficient to prove that broad necessity satisfies 5.

From the δ function we can define another broad notion of necessity: $L'A := \delta A\top\top\bot$ – the connective that outputs $\top$ if A is $\top$ and $\bot$ otherwise. It is straightforward to prove that $L'A \leftrightarrow LA$. If $A = \top$ (i.e. A is L-necessary) then $\delta A\top\top\bot = \top$ by $\delta 1$, since $\top$ is true so is $\delta A\top\top\bot$ by Leibniz's law. This establishes $LA \to L'A$. On the other hand if $\delta A\top\top\bot$ is true then it is not identical to $\bot$ so, by contraposing $\delta 2$, $A = \top$. This establishes $L'A \to LA$. Assuming the stronger principles $L\delta 1$ and $L\delta 2$, it is possible to show that this biconditional

[38] I'm indebted to Peter Fritz here for alerting me of these sorts of correlations between type e and t on the assumption of Kreisel's principle. See also Fritz [18] for some further potential constraints.

[39] Presumably named by analogy with the physicists' Kronecker δ.

[40] In general there is a δ functions at each type $\sigma \to \sigma \to \tau \to \tau \to \tau$.

is L-necessary (here we appeal to standard reasoning involving normal modal operators). In particular every instance of the biconditional is L-necessary, and thus by proposition 2.2, $L'A = LA$.

Given our definition of L', the following are two instances of $\delta 1$ and $\delta 2$:

1. $A = \top \to L'A = \top$
2. $A \neq \top \to L'A = \bot$

In particular the antecedent of 2 is just $\neg LA$ (from the definition of L). The consequent by contrast is equivalent to $\neg L'A = \top$ and by the identity of $L'A$ with LA we have $\neg LA = \top$, or $L\neg LA$. Thus we have an argument for the 5 principle: $\neg LA \to L\neg LA$.

Superficially, it seems as though the above argument does not rest on model theoretic assumptions connecting operators with set-theoretic functions, since it takes the form of a derivation in the object language. However it is nonetheless hard not to suspect that the motivation for the validity of the premises $\delta 1$ and $\delta 2$ rests on the thought that quantification into connective position can be accurately modeled by set-theoretic functions. In particular, it assumes that something like the method of *definition by cases*, something that clearly works with functions, can be applied to operators. Moreover, the only justification we gave for the strengthenings needed in our proof, $L\delta_1$ and $L\delta_2$, was that they are validated in full models, and this justification does seem patently model theoretic.

Indeed it is possible to show that definition by cases, in its strengthened form, is not generally a good way of introducing a predicate or operator.

It is clear that there is a function that takes Alice to the proposition that *it's raining or it isn't* if Alice wins the lottery tomorrow, and to the proposition that *it's raining and not raining* otherwise. But it is hard to think of a predicate that has this behavior. It is even harder to suppose that these identities are necessarily correlated with Alice's winning the lottery, as would be required if we were to secure analogues of $L\delta_1$ and $L\delta_2$. What we would need is a predicate, F, that satisfies the following two principles:

1. $L(A \to (Fa = \top))$
2. $L(\neg A \to (Fa = \bot))$

Here a is Alice and A is the proposition that she wins the lottery tomorrow.

On the assumption that it is contingent whether Alice wins the lottery it is inconsistent to suppose that these two principles are true. For suppose that in fact Alice does win the lottery. Then $Fa = \top$ and so it's L-necessary that $Fa = \top$ by the necessity of identity (which follows from Substitution and the necessity of self identity: $A = B \to L(A = A) \to L(A = B)$). On the other hand it's possible that Alice does not win the lottery. So $M\neg A$, and since we are assuming $L(\neg A \to Fa = \bot)$ it follows, using modal reasoning in K, that $M(Fa = \bot)$. We can put these two things together, again reasoning in K, to conclude that $M(Fa = \top \wedge Fa = \bot)$ and finally $M(\top = \bot)$. Since we can prove that $\top \neq \bot$ we can derive $L(\top \neq \bot)$, a contradiction. The same argument would have worked if we supposed that Alice didn't win the lottery but might have done.

A general principle stating the possibility of definition by cases takes the following form:

> Definition by Cases: for any condition ϕ of type $\sigma \to t$ and propositions B and C there is some X of type $\sigma \to t$ such that, L-necessarily, for any a of type σ, $Xa = B$ if ϕa and $Xa = C$ if $\neg\phi a$.

$\forall Z \forall xy \exists X L \forall z((Zz \to Xz = x) \wedge (\neg Zz \to Xz = y))$ (where X, Z are of type $\sigma \to t$ and x, y, z are of type t).

It is this sort of principle that we would need in order to define things like δ and L' above. But as we have just seen it can be shown to be inconsistent by considering conditions Z that are L-contingent.

The sort of conditions used in the definition of δ were identities. One might hope to patch up the argument by noting that, unlike the condition of winning the lottery, identities are always either L-necessarily false or L-necessarily true. Note, however, that this principle is just another way of stating the necessity of distinctness. As noted earlier, the necessity of distinctness is exactly one of those principle we were attempting to prove (and indeed it is equivalent to the 5 principle), so this version of the argument hardly provides an independent argument for the necessity of distinctness and the 5 principle.

4.4 The Argument from Kripke Models

Another common argument for S5 rests on the idea that the broadest necessity tells us what is true in *all* worlds, and broad possibility what is true in *some* worlds (see, e.g., Lewis [27]). Since our quantifiers are intended to be read unrestrictedly (and not, say, restricted to *accessible worlds*) we might expect the 5 principle to be true: if A is true in some world, then the claim that A is true in some world is true in every world.

But this thought is quite subtle, and rests on a sort of picture thinking — often encouraged by thinking in terms of set-theoretic Kripke models — that is illicit in this context. The objection goes as follows: If w is possible from the perspective of the actual world v, say, but not conversely, then, at w we are not using the modal operator in the broadest sense, since there *is* a possibility, v, which is being excluded. But this is to forget that we are taking seriously the idea that distinct things are possibly identical, and for this very reason we can't assume that an accurate representation of modal reality using indices (representing maximally strong propositions) and an accessibility relation (representing relative possibility) has a modally rigid structure. The picture I envisage is that the proposition that corresponds to the actual world — represented by the set $\{v\}$ in our model — while distinct from the inconsistent proposition — represented by the empty set — *would* be identical to it had w obtained. Presumably this is nonsense when we are talking about the representations of the propositions in a Kripke model: the empty set is necessarily distinct from any singleton set. But in that case, propositions cannot safely be assumed to be sets of any sort, since we wish to take seriously the hypothesis that propositions can be contingently distinct.

It is, in fact, possible to formulate the argument in higher-order logic by understanding worlds as certain kinds of maximally strong propositions which we call *world propositions* — a proposition which is both broadly possible and broadly entails every proposition or its negation — and by understanding a proposition being *true at* a world in terms of the world strictly implying it in the broad sense (see Fine [16]). The above argument for the 5 principle then amounts to the claim that if a there is a world proposition W that entails A, then every world proposition entails that there is a world proposition that entails A. It is a theorem of our logic that every proposition necessarily exists, and so in particular W necessarily exists. Moreover if W broadly entails a proposition, it necessarily entails it — an instance of the 4 principle, $L(W \to A) \to LL(W \to A)$. A natural thought would be that since it's necessary that W entails A, it follows that it's necessary that there's a world proposition (namely W) that entails A. But this reasoning works only if we had some argument that W is *necessarily* a world proposition. What if W is in fact a world proposition but fails to be a world proposition at at the envisaged worlds at which A is impossible. Recall, in particular, that a world proposition must be broadly possible. A proposition is broadly possible if it is distinct from $\bot$. But if we are taking contingent distinctness seriously, and our proposition is only contingently distinct from $\bot$, then a proposition that's in fact possible (distinct from $\bot$) could possibly be impossible (identical to $\bot$). Since we have no guarantee that world propositions are necessarily world propositions, the proposed argument for S5 fails, even when we understand the quantifiers as ranging unrestrictedly.

5 Conclusion

We have raised some considerations in favor of and against the validity of the theorems of S5 for broad necessity, and have made a tentative case for a weaker logic.

Of course, one source of reluctance about giving up S5 is that it is a relatively simple and well understood system, and it is moreover familiar because it is commonly held to be the logic of metaphysical necessity. It is important to note that even if the broadest necessity is not governed by a logic as strong as S5, it is completely consistent to assume that weaker modalities, such as metaphysical modality, are.

It is possible, at any rate, that our expectation that the broadest necessity abide by a logic of S5 is a holdover from the assumption that metaphysical necessity is the broadest necessity. Once we have acknowledged the non-trivial interaction between metaphysical necessity and vagueness, for example, we both have reasons to think that metaphysical necessity is not the broadest necessity, and that the broadest necessity has a weaker logic. The fact that S4 (and not S5) is the most one can derive from a natural axiomatisation of higher-order logic is also quite suggestive in this regard.

Of course, having shown that S5 is not forced upon us or that it is in fact invalid, doesn't settle what the logic of broad necessity actually is. For all we've said it could be a stronger logic containing S4 but not properly extending S5. One might hope to settle this question by an analogue of Scroggs' argument, however this seems unlikely as the logics extending S4 do not have a straightforward characterization (see Dummett and Lemmon [12]), and the principles involved are far more contentious. We leave these questions for further work.

6 Appendix

6.1 Models of Higher-Order Logic

We work within the simply typed λ-calculus with one base type, t. All other types may be obtained as follows: t is a type, and if σ and τ are types, so is $\sigma \to \tau$.[41]

Type signatures such as the one described above can in general be modeled by *applicative structures* (see Mitchell [30]). Here we shall focus on a particular kind of applicative structure:

> A *Henkin structure* consists of a collection of sets A^σ indexed by the types σ with the property that

[41] We have excluded the type e since all of the relevant definitions involve types constructed only from t. This makes the presentation simpler, although nothing turns on this.

- $A^{\sigma\to\tau} \subseteq A^{\tau A^{\sigma}}$ for each σ and τ

Henkin structures then assign meanings to all type expressions. In general Henkin structures are too impoverished too interpret higher-order logic: we need to ensure that they contain enough functions to interpret the typed λ-calculus.

> A Henkin structure is *rich* iff, for each types σ, τ, υ there are elements $K_{\sigma\tau} \in A^{\sigma\to\tau\to\sigma}$ and $S_{\sigma\tau\upsilon} \in A^{(\sigma\to\tau\to\upsilon)\to(\sigma\to\tau)\to\sigma\to\upsilon}$ satisfying the following properties:
>
> - $K_{\sigma\tau}xy = x$ whenever $x \in A^{\sigma}$ and $y \in A^{\tau}$
> - $S_{\sigma\tau\upsilon}xyz = xz(yz)$ whenever $x \in A^{\sigma\to\tau\to\upsilon}$, $y \in A^{\sigma\to\tau}$ and $z \in A^{\sigma}$

Note that any model of the simply typed λ-calculus must contain S and K in each relevant type, because we can define functions with their behavior just using λ-terms: $\lambda x\lambda y\, x$ and $\lambda x\lambda y\lambda z\, xz(yz)$. Of more note is the fact that if a Henkin structure contains S and K then it contains every λ-definable function (see Mitchell [30] chapter 4).

A signature Σ for a simply typed λ-calculus consists of a set of constants c and a type assignment function Typ mapping each constant to a type. Given an infinite set of variables Var, the type assignment function can be expanded so that it surjectively assigns types to every member of Var in such a way that the preimage of each type is infinite. We then expand Typ to assign types to strings of symbols from our signature as follows (below and elsewhere we write 'α has type σ' when $Typ(\alpha) = \sigma$):

> c has type $Typ(c)$
>
> x has type $Typ(x)$
>
> $\alpha\beta$ has type τ when α has type $\sigma \to \tau$ and β has type σ
>
> $\lambda x\, \alpha$ has type $\sigma \to \tau$ when x has type σ and α type τ

A term of the simply typed λ-calculus of signature Σ, $\mathcal{L}_{\Sigma}$, is any string in the domain of Typ.

A variable assignment is a function g on Var such that $g(x) \in A^{Typ(x)}$ for each $x \in Var$. We write $g[x \mapsto d]$ for the assignment that is exactly like g except it maps x to d. If a Henkin structure is rich it is possible to interpret the simply typed λ-calculus over a given signature.

> A *Henkin model* of a signature Σ is a pair $(A, [\![\,]\!])$ where A is a Henkin structure and $[\![\cdot]\!]^{\cdot}$ a function taking each term of $\mathcal{L}_{\Sigma}$ of type σ and variable assignment to an element of A^{σ}, satisfying the following properties:
>
> 1. $[\![c_{\sigma}]\!]^{g} \in A^{\sigma}$
> 2. $[\![x]\!]^{g} = g(x)$
> 3. $[\![\alpha\beta]\!]^{g} = [\![\alpha]\!]^{g}([\![\beta]\!]^{g})$
> 4. $[\![\lambda x\alpha]\!]^{g} =$ the unique function $f \in A^{\sigma\to\tau}$ such that $f(d) = [\![\alpha]\!]^{g[x\mapsto d]}$ for every $d \in A^{\sigma}$

Notice that if there is a function satisfying the condition in the last clause, it is unique by the functionality of Henkin models. The fact that we have required the Henkin model to be rich guarantees that there is always at least one function that satisfies the condition; this follows from the point, noted above, that a rich Henkin structure contains every λ-definable function.[42]

To interpret higher-order logic we focus on the signature $H = \{\to\} \cup \{\forall_\sigma \mid \sigma \in Type\}$ where $\to$ has type $t \to t \to t$ and $\forall_\sigma$ type $(\sigma \to t) \to t$.

> A *logical* Henkin model is a a triple $(A, [\![\,]\!], T)$ where $(A, [\![\,]\!])$ is a Henkin model, $\emptyset \subset T \subset A^t$, and moreover,
>
> – $[\![\to]\!](a)(b) \in T$ iff $a \notin T$ or $b \in T$
>
> – $[\![\forall_\sigma]\!](f) \in T$ iff $f(a) \in T$ for every $a \in A^\sigma$.
>
> Term ϕ of type t is *true* in a logical model M if and only if $[\![\phi]\!]^g \in T$ for every assignment g.

It is easily verified that every theorem of H is true in a logical Henkin model.

Suppose A is a rich Henkin structure, and that A^t is a Boolean algebra such that every subset of A^t which is the range of some function $f \in A^{\sigma \to t}$ has a conjunction in A^t. (This is satisfied, in particular, if A^t is a *complete* Boolean algebra.) We may then define a logical Henkin model as follows. Let $[\![\forall_\sigma]\!](f) = \bigwedge_{a \in A^\sigma} f(a)$ and $[\![\to]\!](a)(b) = \neg a \vee b$ (where $\neg, \vee$ and $\bigwedge$ express the Boolean operations), and let T be an ultrafilter on A^t. It is easily verified that $(A, [\![\,]\!], T)$ is a logical Henkin model, that moreover makes all of the theorems of HE true. The status of the functionality principle is more subtle. In a Henkin model, but not in an arbitrary applicative structure, if two elements $f, g \in A^{\sigma \to \tau}$ output the same thing for every argument, they are identical. This ensures that we can close our theory under a weak functionality rule: if $\vdash \phi x = \psi x$ then $\vdash \phi = \psi$. However there are logical Henkin models in which the functionality axiom of section 3 is false; we will attend to this matter later in 6.4.

6.2 Kripke Logical Relations

We now introduce an important definition from Plotkin [32].

> Let A be a Henkin structure, and let (W, R) be a reflexive transitive Kripke frame. A binary *Kripke logical relation* over (W, R) consists in, for each $x \in W$, a typed family of binary relations $\sim^x_\sigma$ over A^σ with the following properties.
>
> – For every $a, b \in A^\sigma$, if $a \sim^\sigma_x b$ and Rxy then $a \sim^\sigma_y b$.
>
> – For $f, g \in A^{\sigma \to \tau}$, $f \sim^{\sigma \to \tau}_x g$ if and only if $fa \sim^\tau_y gb$ for every y such that Rxy and $a, b \in A^\sigma$ such that $a \sim^\sigma_y b$.

Clearly once you have fixed the behavior of a Kripke logical relation on the base domain A^t its behavior is determined on all higher types. There is a close connection between Kripke logical relations and the Kripke semantics for intuitionistic logic, in which we talk of types being *true* at worlds, and in which a functional type $\sigma \to \tau$ is true at a world x only if τ is true at every σ world that is R-accessible from x. For convenience we shall continue to

[42] Note that if we are working in an applicative structure that is not functional, then further constraints on S and K are needed to to ensure that the λ-terms obey $\beta\eta$ conversion. See Hindley and Seldin [21]p 86.

suppress the type superscripts when the types can be inferred from the context. In what follows we will restrict attention to a particular Kripke logical relation generated by a family of equivalence relations $\sim_x^t$ on the base type; in general, however, the generating relation for a Kripke logical relations need not be equivalence relation (and even if a Kripke logical relation is generated from equivalence relation on the base types, it does not follow that it is an equivalence relation on the higher types). The notion of a Kripke logical relation makes sense for higher arity relations as well.

Kripke logical relations where originally introduced by Plotkin in order to give a characterization of the λ-definable functions in a model of type theory. For our purposes, the most important result concerning Kripke logical relations is that every λ-definable function is invariant under every Kripke logical relation.

Theorem 6.1 (Plotkin)**.** *Let $(A, [\![\,]\!])$ be a Henkin model of the empty signature, (W, R) a transitive Kripke frame, and and $\sim$ a Kripke logical relation on A with respect to (W, R). Then for every closed term α, and world $x \in W$, $[\![\alpha]\!] \sim_x [\![\alpha]\!]$.*

Proof. Given two variable assignments g and h we say that $g \sim_x h$ if $g(v) \sim_x h(v)$ for every variable $v \in Var$.

We will prove, by induction on term complexity, a stronger hypothesis that if $g \sim_x h$ then $[\![\alpha]\!]^g \sim_x [\![\alpha]\!]^h$. The hypothesis is clearly true for variables.

Suppose that it is true for α and β of types $\sigma \to \tau$ and σ respectively, and suppose that $g \sim_x h$. Then $[\![\alpha]\!]^g \sim_x [\![\alpha]\!]^h$ by inductive hypothesis. By the definition of a Kripke logical relation, that means that if Rxy and $a \sim_y b$, $[\![\alpha]\!]^g(a) \sim_y [\![\beta]\!]^h(b)$. In particular, since $[\![\beta]\!]^g \sim_x [\![\beta]\!]^h$ by inductive hypothesis, and Rxx, if follows that $[\![\alpha]\!]^g([\![\beta]\!]^g) \sim_x [\![\alpha]\!]^h([\![\beta]\!]^h)$. Thus $[\![\alpha\beta]\!]^g \sim_x [\![\alpha\beta]\!]^h$ as required.

Now suppose that the hypothesis is true for α of type τ, let v be a variable of type σ and suppose $g \sim_x h$. We wish to show that $[\![\lambda v\alpha]\!]^g \sim_x [\![\lambda v\alpha]\!]^h$, so suppose that Rxy and $a \sim_y b$. Then $g[v \mapsto a] \sim_y h[v \mapsto b]$, and thus by inductive hypothesis $[\![\alpha]\!]^{g[v\mapsto a]} \sim_x [\![\alpha]\!]^{h[v\mapsto b]}$. This is just to say that $[\![\lambda v\alpha]\!]^g(a) \sim_x [\![\lambda v\alpha]\!]^h(b)$, as required for the equivalence of functions. □

The restriction to the empty signature can be lifted if we additionally impose that $[\![c_\sigma]\!] \sim_x^\sigma [\![c_\sigma]\!]$ for each constant c_σ in the language.

6.3 Completeness of S4 in models of HOL

Given any transitive reflexive frame (W, R) we define a Henkin structure using Kripke logical relations. We define A^σ and $\sim_x^\sigma$ simultaneously by induction. Here we write $R(x)$ to abbreviate $\{y \in W \mid Rxy\}$.

To define A^t and $\sim_\cdot^t$:

- $A^t = \mathcal{P}(W)$
- $p \sim_x^t q$ iff $p \cap R(x) = q \cap R(x)$.

To define $A^{\sigma\to\tau}$ and $\sim_\cdot^{\sigma\to\tau}$

- Suppose $f, g : A^\sigma \to A^\tau$. We define $f \sim_x^{\sigma\to\tau} g$ iff for every y st Rxy, and $a, b \in A^\sigma$ st $a \sim_y^\sigma b$, $f(a) \sim_y^\tau g(b)$.
- $A^{\sigma\to\tau} = \{f : A^\sigma \to A^\tau \mid f \sim_x f \text{ for every } x \in W\}$.

Note that although $\sim_x^{\sigma\to\tau}$ is a relation on ${A^\tau}^{A^\sigma}$, it can also be considered a relation on $A^{\sigma\to\tau}$ by restriction.

We may obtain a logical Henkin model M from A by selecting some world $@ \in W$ as the actual world:

$p \in T$ if and only if $@ \in p$.

$[\![\forall_\sigma]\!](f) = \bigcap_{a\in A^\sigma} f(a)$

$[\![\to]\!](p,q) = (W \setminus p) \cup q$

To show that M really is a logical Henkin model we must show that M is rich, and moreover contains the interpretations of $\forall_\sigma$ and $\to$ above.

Proposition 6.2. *M is a model of* HE*.*

Proof. For each type σ, τ we defined $\sim$ on the full set of functions ${A^\tau}^{A^\sigma}$. However when $\sim$ is appropriately restricted to $A^{\sigma\to\tau}$ it is easy to see that it is a Kripke logical relation on our model M. It follows by theorem 6.1 that $S_{\sigma\tau\upsilon}$ and $K_{\sigma\tau}$ are invariant under $\sim$, and so both belong to M.

It remains to show that the interpretations of $\forall_\sigma$ and $\to$ are in M. For $\forall_\sigma$, suppose $f, g \in M_{(\sigma\to t)\to t}$ and $f \sim_x g$. For each $a \in M_\sigma$, $a \sim_x a$ so $fa \sim_x ga$. Expanding the definition of $\sim_x$, this means that $R(x) \cap fa = R(x) \cap ga$ for each a, and so $R(x) \cap \bigcap_{a\in M_\sigma} fa = R(x) \cap \bigcap_{a\in M_\sigma} ga$. That is to say $[\![\forall_\sigma]\!](f) \sim_x [\![\forall_\sigma]\!](g)$.

M is thus a logical Henkin model, in which A^t is a complete Boolean algebra, T and ultrafilter on A^t, as described earlier. Thus M is a model of HE. □

The next proposition shows that propositional identity in our model, which is defined by Leibniz equivalence — $\forall X(Xp \leftrightarrow Xq)$ — amounts to the same thing as necessary equivalence relative to the modality defined by the accessibility relation R. In particular $x \in [\![\forall X(Xp \leftrightarrow Xq)]\!]$ if and only if every world accessible to x belongs to $[\![p \leftrightarrow q]\!]$. This also has the consequence that $x \in [\![LA]\!]$ if and only if $y \in [\![A]\!]$ for every y such that Rxy — that is, L is governed by a standard Kripke semantics in terms of the accessibility relation R.

Proposition 6.3. *Let $p, q \in A^t$. Then for each $x \in W$, $p \sim_x q$ if and only if, for every $f \in A^{t\to t}$, $x \in f(p) \Leftrightarrow x \in f(q)$. In other words, Leibniz equivalence corresponds to being necessarily equivalent (relative to R) in our model.*

Proof. Suppose that $p \sim_x q$ and let $f \in A^{t\to t}$. Since f preserves $\sim_x$ it follows that $f(p) \sim_x f(q)$ — i.e. $f(p) \cap R(x) = f(q) \cap R(x)$. Since R is reflexive, $x \in R(x)$ and so $x \in f(p)$ if and only if $x \in f(q)$.

Conversely suppose that $p \not\sim_x q$. Define a function f as follows: $f(X) := \{y \mid X \sim_y p\}$. Clearly $x \in f(p)$ and $x \notin f(q)$. It remains to show that $f \in A^{t\to t}$ — i.e. that f preserves $\sim_z$ for each world z.

Suppose, then, that $X \sim_z Y$. We want to show that $f(X) \sim_z f(Y)$: that every $f(X)$ world accessible to z is an $f(Y)$ world and conversely. Let Rzw suppose that $w \in f(X)$. By the definition of X that means $X \sim_w p$. Since $X \sim_z Y$, $X \sim_w Y$ and since $\sim_w$ is an equivalence relation $Y \sim_w p$. So $w \in f(Y)$. The converse direction proceeds in exactly the same way, so $f(X) \sim_z f(Y)$. □

Corollary 6.4. *The functionality principle is true in M.*

Proof. For any given world, x we want to show that $x \in [\![\forall x(Fx = Gx) \to F = G]\!]$, recalling again that $=$ is short for Leibniz equivalence.

By proposition 6.3 it suffices to show that if $f, g \in M_{\sigma\to\tau}$ and $fa \sim_x ga$ for every $a \in M_\sigma$ then $f \sim_x g$, since we have $\sim_x$ corresponds to Leibniz equivalence in our model. So suppose the hypothesis, and let $a \sim_x b$. Since f and g are in M they preserve $\sim_x$, and so $fa \sim_x fb$ and $ga \sim_x gb$. Since $\sim_x$ is an equivalence relation, $fa \sim_x gb$, and since this holds for every such a and b, $f \sim_x g$ as required. □

Theorem 4.1 follows from the above as a corollary. Suppose that $(W, R, [\![\,]\!])$ is an transitive reflexive Kripke model (see Hughes and Cresswell [22]) and that $@ \in W$. Then we may construct a logical Henkin model $(A, [\![\,]\!]', T)$ from (W, R) and $@$ as above, setting $[\![p]\!]' := [\![p]\!]$ for each propositional constant p. By proposition 6.3 we know that $w \in [\![LA]\!]'$ iff $x \in [\![A]\!]$ for every x such that Rwx, provided $[\![A]\!]' = [\![A]\!]$, and so by a simple induction we can show that for any formula θ of the modal language $\mathcal{L}_L$ (defined in section 3), $w \in [\![\theta]\!]'$ if and only if $w \in [\![\theta]\!]$.

Now suppose that ϕ is not provable in $\mathsf{S4}$. Then ϕ is false at some world $@$ in some transitive Kripke model (W, R) and thus it is false in some model of HFE. Thus ϕ is not provable in HFE. Conversely, we have shown in section 3 that every theorem of $\mathsf{S4}$ is provable in HFE.

The sorts of models described above can be used to show other independence results. We end by briefly describing how to use these techniques to construct a model (mentioned in section 4.2) in which there is a bijection of type $e \to t$ with no inverse in $t \to e$

> Let $A^e = A^t = P(\mathbb{N})$.
>
> For $x, y \in \mathbb{N}$ let Rxy iff $x \leq y$. (Any preorder that isn't an equivalence relation would work here.)

Now we define a world indexed equivalence relation and domains simultaneously as above.

> For $a, b \in A^e$, $a \sim_w b$ iff $a = b$
>
> For $a, b \in A^t$, $a \sim_w b$ iff $R(w) \cap a = R(w) \cap b$.
>
> For $f, g \in A^{\sigma\to\tau}$, $f \sim_w g$ iff for every x such that Rwx, and every $a, b \in A^\sigma$ such that $a \sim_x b$, $fa \sim_x gb$.
>
> $A^{a\to b} = \{f : A^a \to A^\tau \mid \text{for every } w, f \sim_w f\}$

By setting $@ := 0$ we can define a logical Henkin model as above which makes all of the theorems of HFE true.

It is immediate that any bijective function $f : A^e \to A^t$ preserves $\sim_x$ for each world x and thus that $f \sim_x f$, because $\sim_x^e$ is just identity. So $f \in A^{e\to t}$. It should be noted that two propositions $a, b \in A^t$ are Leibniz equivalent at 0 ($0 \in [\![x = y]\!]^{g[x\mapsto a, y\mapsto b]}$) if and only if $a = b$, since by proposition 6.3 a and b are Leibniz equivalent at 0 iff they are necessarily equivalent at 0. It is then easy to verify that a function f is satisfies the the object language statement that f is a bijective concept of type $e \to t$ at 0, iff f is in fact a bijective function.

Note that bijections are at best contingently bijections. A bijection from A^e to A^t does not count as 'bijective' at any world > 0. Let $a = \mathbb{N}$ and $b = \mathbb{N} \setminus \{0\}$. Then $a \sim_1 b$, but since f is a bijection $f(a) \neq f(b)$ so $f(a)$ and $f(b)$ are not identical at 1 in A^e. f thus fails to be injective at 1 because it takes distinct things at 1 of type e to identical things at 1 of type t.

This example also shows that no bijection can belong to $A^{t\to e}$, since to belong to this domain you must preserve $\sim_1$, yet $a \sim_1^t b$ but for any bijection $fa \neq fb$ (since a and b are distinct) and so $fa \not\sim_1^e fb$. Since no bijective functions belong to $A^{t\to e}$, the claim that there's a bijective function from $t \to e$ is false at 0 (because, as noted above, a function counts as bijective at 0 iff it's a bijection.) The object language claim 'no bijection of type $e \to t$ has an inverse' is also true in this model for similar reasons.

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